Supporting Information for *Fractal Patterns in the Parameter Space of Bi-stable Duffing Oscillator*

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Contents

S1 Nondimensionalization of Bi-stable Duffing Oscillator Equation

The harmonically excited bi-stable system undergoes exotic dynamics. A variety of behaviors including switching, reversion, vacillation, and intra-well may happen under single-frequency harmonic excitation.

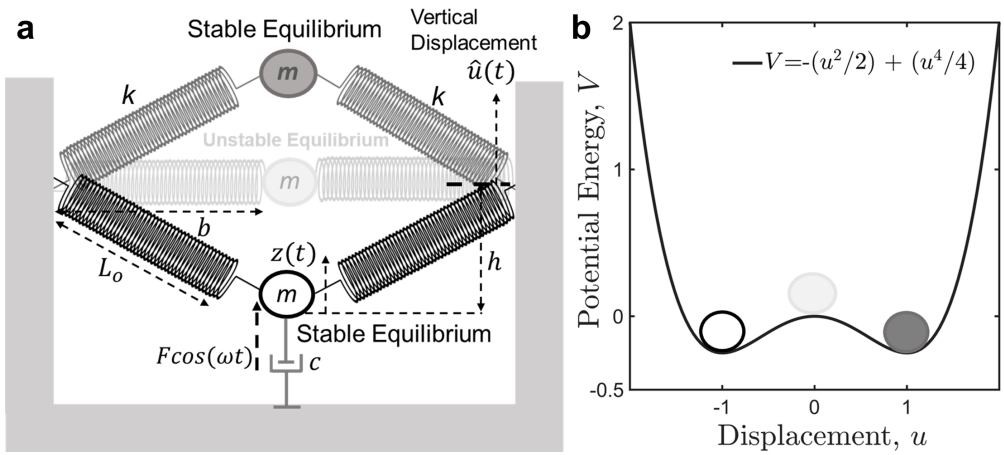


Figure S1: An illustration of a bi-stable duffing system with two equal energy wells described in the main text.

As an example of physical realization, Fig. S1(a) shows a single degree of freedom system consisting of a point mass, m under harmonic excitation $F\cos(\omega t)$ with viscous damping, c, and two linear stiffness, k. We denote the vertical displacement as $z(t)$ and the force in each spring as F_s, which has horizontal and vertical components F_h and F_0, respectively, as shown in Fig. S2. While the system has a total width of $2b$, each linear spring has the zero-force

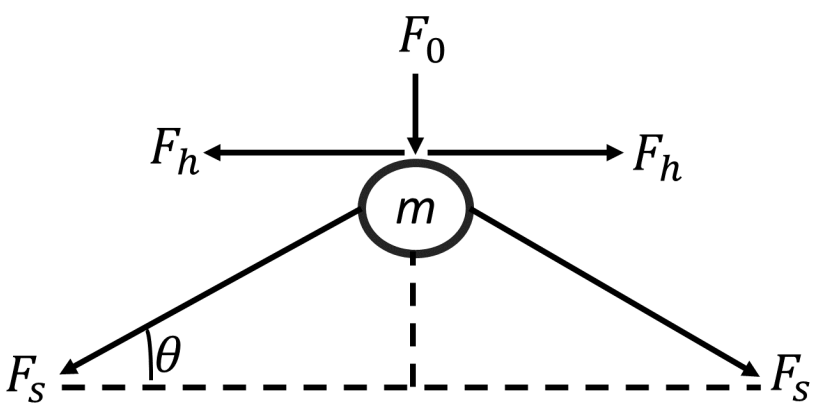


Figure S2: A schematic of bistable spring restoring force

equilibrium length $L_0 > b$. We consider the state where the springs are at an angle θ with the horizontal direction, and the vertical component of the spring force, F_0, is denoted by,

$$F_0 = 2k(L_0 - L)\sin(\theta) \tag{S1}$$

Where $\sin(\theta) = \frac{h-z(t)}{L}$. $L_0 = \sqrt{h^2+b^2}$ is the unstretched length of the spring, and $L = \sqrt{(h-z(t))^2+b^2}$ is the stretched length of the spring. Hence, we have,

$$F_0 = 2k(h-z)\left(\frac{\sqrt{h^2+b^2}}{\sqrt{(h-z)^2+b^2}} - 1\right) \tag{S2}$$

Now, let $\hat{u}(t) = h - z(t)$, and then the Eq. (S2) becomes,

$$F_0 = 2k\hat{u}\left(\frac{\sqrt{h^2+b^2}}{\sqrt{\hat{u}^2+b^2}} - 1\right) \tag{S3}$$

$$F_0 = 2k\hat{u}\left(\frac{L_0}{\sqrt{\hat{u}^2+b^2}} - 1\right) \tag{S4}$$

Assuming $\frac{\hat{u}(t)}{b} << 1$, expanding Eq. (S4) using Taylor series expansion yields (higher order terms omitted),

$$F_0 = 2k\left(\frac{L_0}{b} - 1\right)\hat{u} - \frac{kL_0}{b^3}\hat{u}^3 \tag{S5}$$

Thus, we have the stiffness term of the system in the polynomial form $-k_1\hat{u} + k_3\hat{u}^3$, where the coefficients are $k_1 = 2k\left(\frac{L_0}{b} - 1\right)$ and $k_3 = \frac{kL_0}{b^3}$.

Now, the bistable spring is excited under harmonic excitation $F\cos(\omega t)$, dynamic governing equation of the bi-stable spring-mass system is represented by the below equation,

$$m\frac{d^2\hat{u}}{dt^2} + c\frac{d\hat{u}}{dt} - k_1\hat{u} + k_3\hat{u}^3 = F\cos(\omega t). \tag{S6}$$

Below, we show the complete nondimensionalization of Eq. (S6): To start, the non-dimensional length and time can be defined as $u = \hat{u}/l$ and $\tau = t\omega_{\mathrm{n}}$, where $\omega_{\mathrm{n}} = \sqrt{\frac{k_1}{m}}$ is the natural frequency at the linear limit. Additional dimensionless parameters are $\gamma = \frac{c}{m\omega_n}$, $\mu = \frac{k_3 d^2}{m\omega_n^2}$, $\Omega = \frac{\omega}{\omega_n}$, and $G = \frac{F}{l\omega_n^2}$. Here, the characteristics length is chosen in such a way that $l = \sqrt{k_1/k_3} \Rightarrow \mu = 1$. After non-dimensionalization, Eq. (S6) becomes

$$\ddot{u} + \gamma\dot{u} - u + u^3 = G\cos(\Omega\tau) \quad \text{with} \quad \ddot{u} = \frac{d^2u}{d\tau^2} \quad \text{and} \quad \dot{u} = \frac{du}{d\tau} \tag{S7}$$

For $\mu = 1$, Eq. (S7) results in the symmetric energy potential shown in Fig. S1(b). If we choose the characteristic length in such a way that $l_c = \sqrt{k_1/k_3} \Rightarrow \mu = 1$, the potential landscape has double symmetric wells around two stable equilibria at $u_{-1} = -1$ and $u_{+1} = +1$, which are separated by an unstable hilltop equilibrium at $u_0 = 0$.

S2 Static Loading Required to Switch between Stable States

Under the quasi-static loading condition, Eq. (S7) becomes

$$-u + u^3 = F_{\text{static}}. \tag{S8}$$

Next, by solving $dF_{\text{static}}/du = 0$, we obtain the critical displacement $u^* = \pm\sqrt{3}/3$, which results in the dimensionless critical force amplitude for switching, $F^*_{\text{static}} = |-u^* + (u^*)^3|$. This shows the dimensionless force required is $F^*_{\text{static}} = 2\sqrt{3}/9 \approx 0.38$ to move between the two stable states, as depicted in Fig. S3.

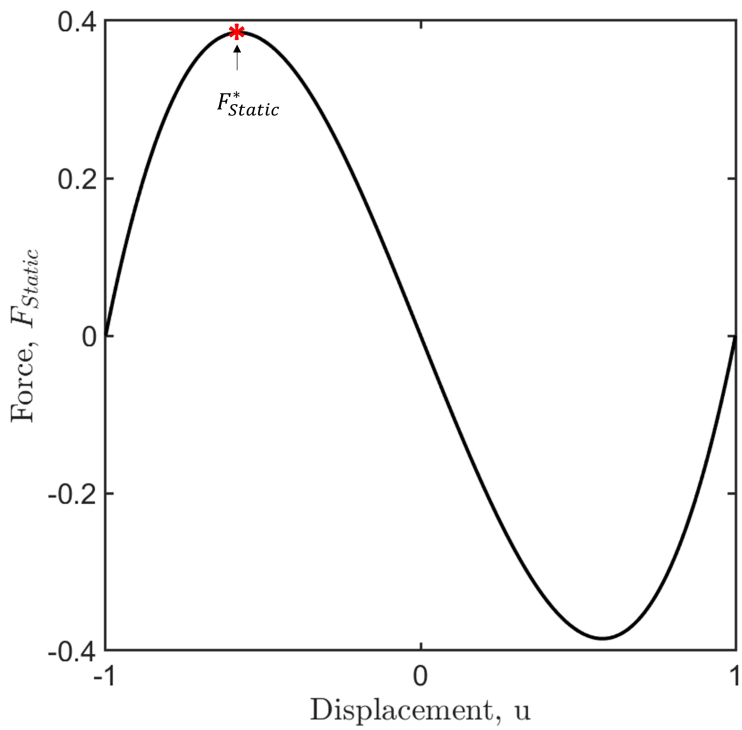


Figure S3: The force-displacement response of the bistable system in the static limit.

S3 Box-Counting Algorithm for Calculating Fractal Dimension

Box-counting algorithm for calculating the fractal dimension of the boundary between inter-well and intra-well motion is shown below. We execute the following step to calculate the

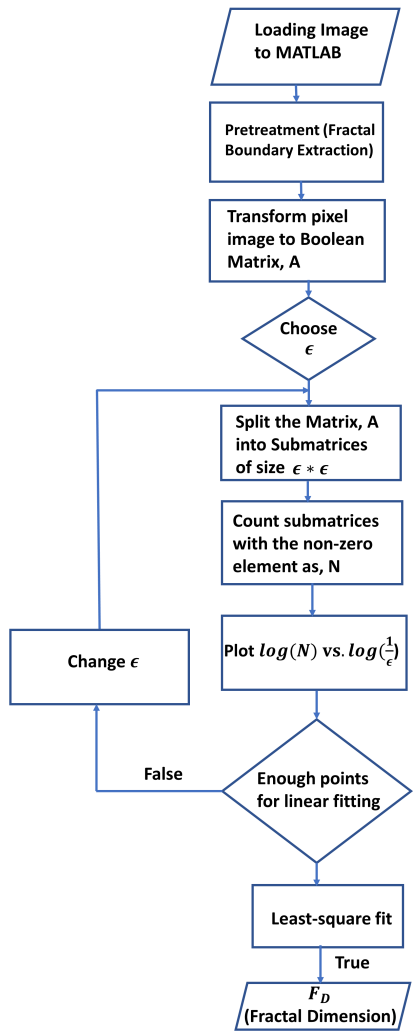


Figure S4: An algorithm for calculating the fractal dimension, F_D

F_{D} of Figs.5(b) and 5(d) in the main text. First, we pre-treat Figs.5(a) and 5(c) in the main text by utilizing the image processing toolbox of MATLAB and eroding the gray-scale image to get the boundary between inter-well and intra-well motion. Then, we extract the binary image to get the pixel data corresponding to the Boolean matrix (contains 1 and 0) and split the Boolean matrix into sub-matrices of $\epsilon * \epsilon$ in the sequence of ($1 < \epsilon <$ smaller side length of the image). Afterward, we count N boxes of non-zero element matrix according to ϵ size. Using the least-square method, we linearly fit the data $\log(N)$ vs. $\log(\frac{1}{\epsilon})$ plot. Then, the absolute value of the slope of Figs.5(e) and (f) in the main text yields a

fractal dimension F_{D}. When performing the box-counting algorithm to calculate the fractal dimension (e.g., Fig. 5(a) and (d) in the main text), we run additional simulations and magnify the parameter space to different levels of pixel resolutions. Then, we calculate the fractal dimension at each level until it converges. Fig. S5 shows an example of the process for $\gamma = 0.001$. We observe that the fractal dimension initially rises with increasing resolution but eventually reaches a plateau where it is stabilized. This plateau indicates our numerical algorithm has led to convergence, and further magnification does not result in significant changes to the fractal dimension calculation.

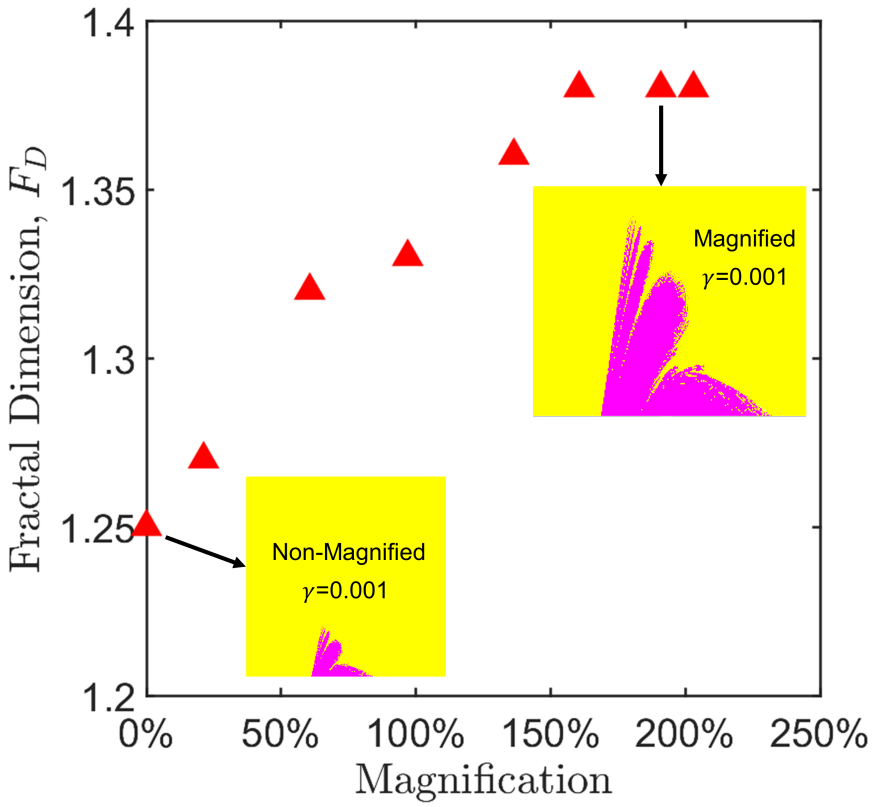
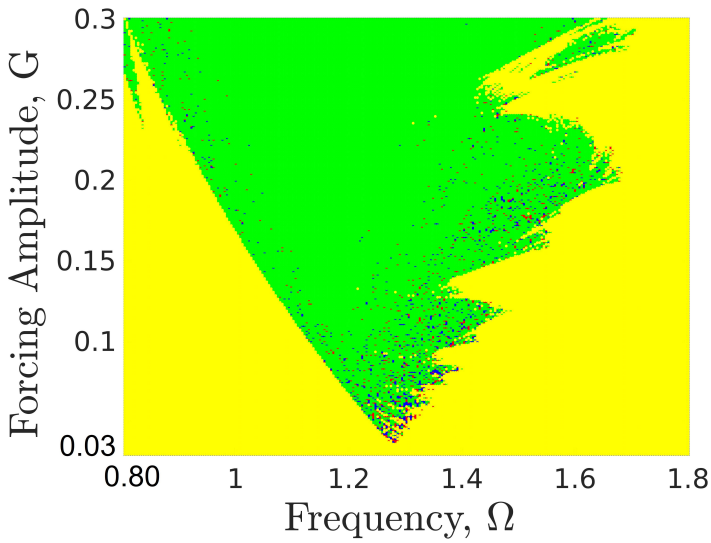


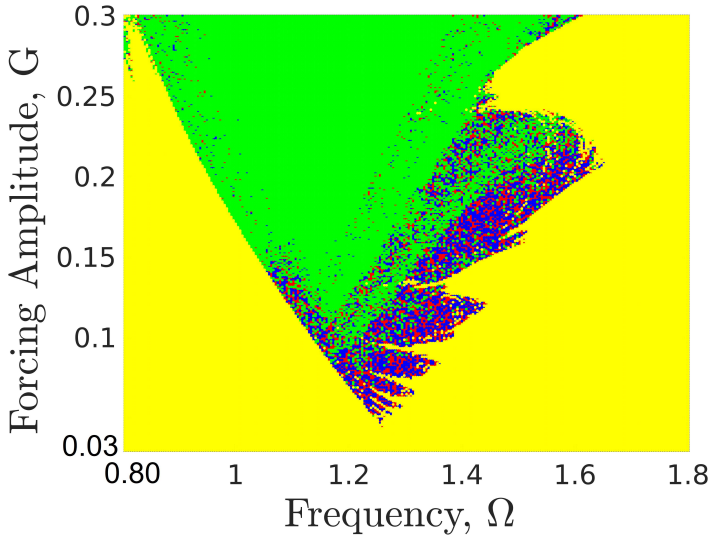
Figure S5: Fractal dimension of the parameter space for $\gamma = 0.001$ as a function of magnification level. The fractal dimension increases with increasing magnification level but eventually reaches a plateau where it stabilizes, indicating that the fractal pattern is self-similar at that level of magnification.

S4 Forcing Amplitude-Frequency (G vs. Ω) Parameter Space

31 forcing amplitude-frequency parameter space for γ ranges from $0.001 \sim 0.30$ with the increment of 0.01.

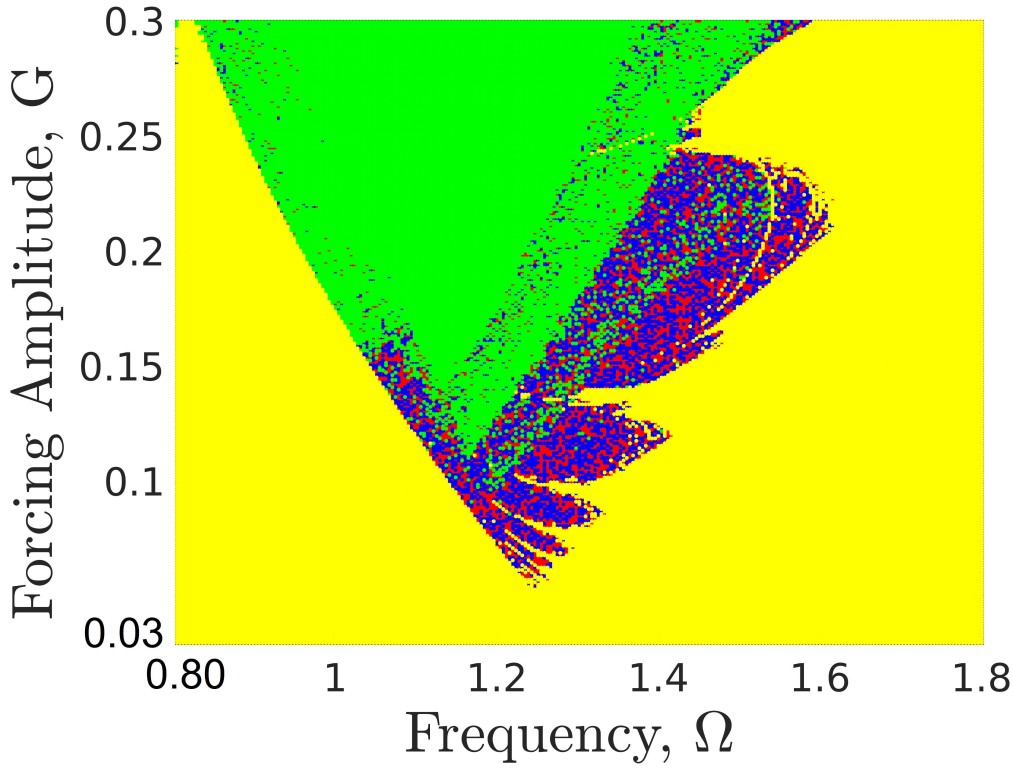


(a) $\gamma = 0.001$

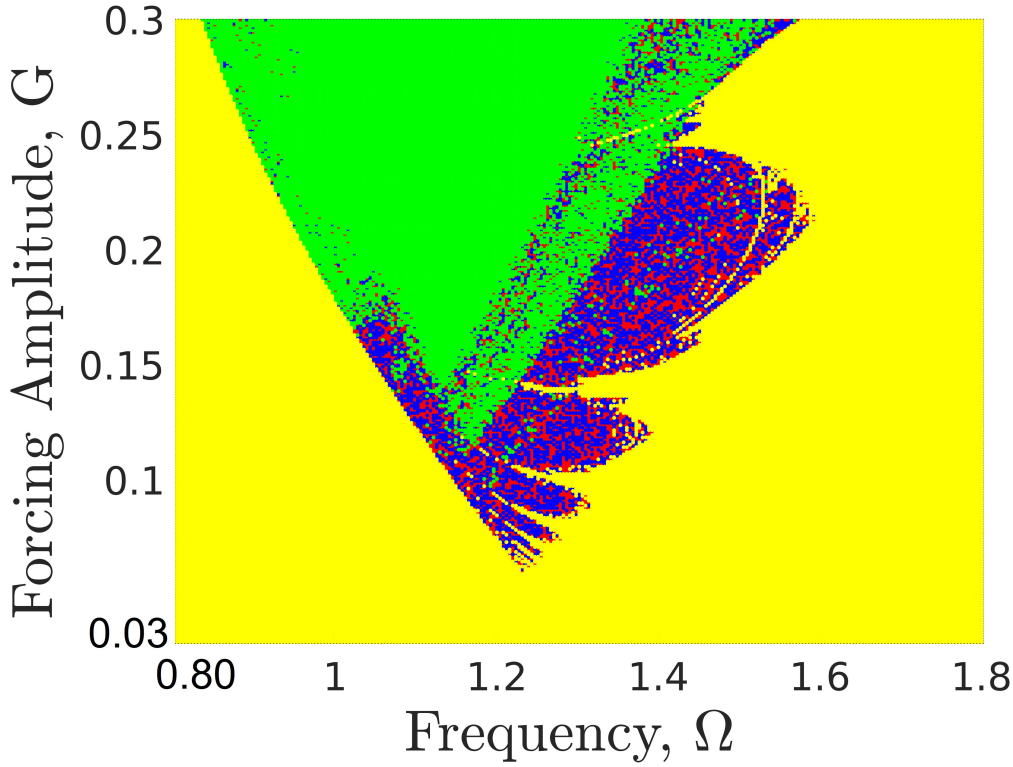


(b) $\gamma = 0.01$

Figure S6: Forcing amplitude-frequency parameter space

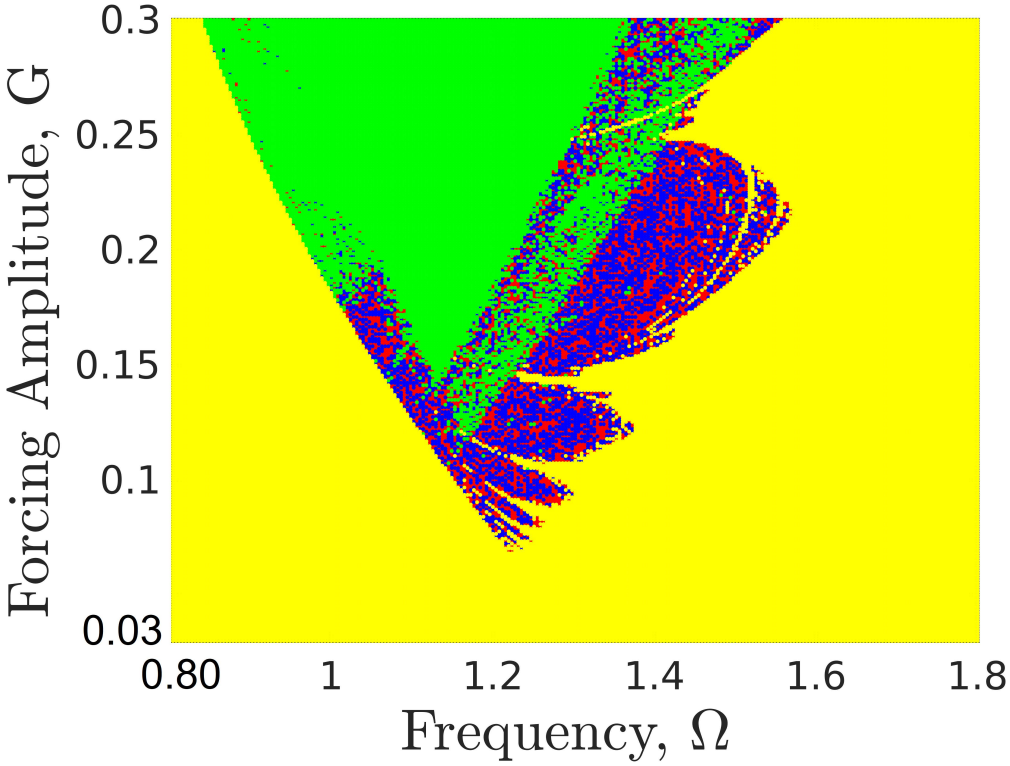


(c) $\gamma = 0.02$

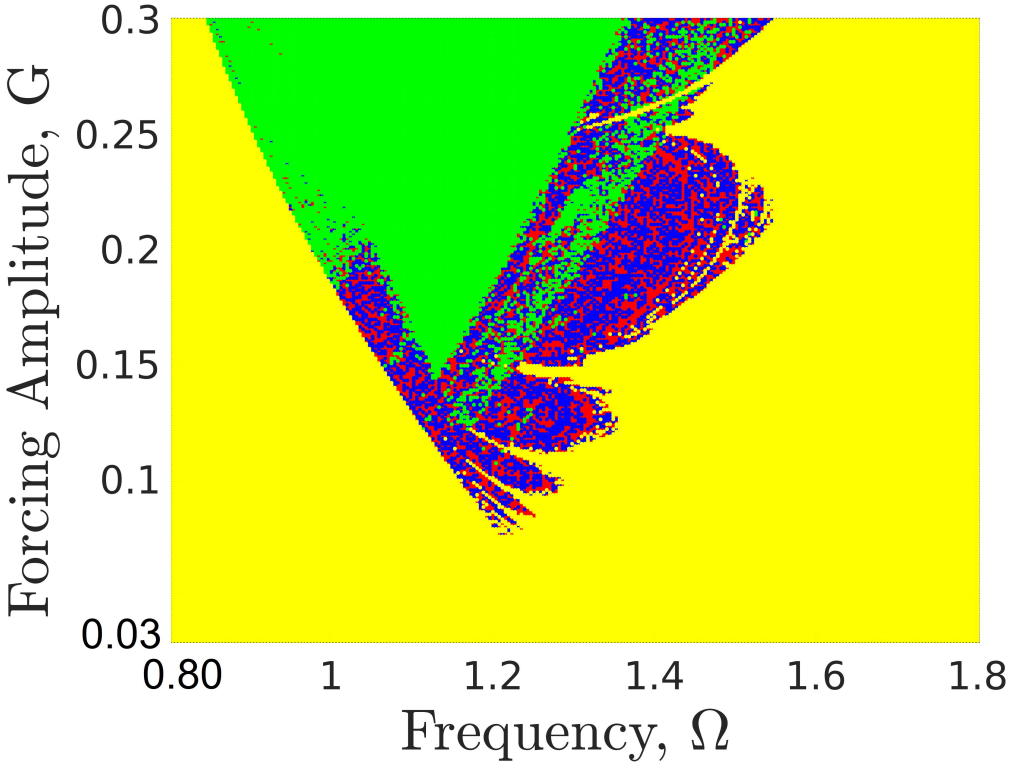


(d) $\gamma = 0.03$

Figure S6: Forcing amplitude-frequency parameter space

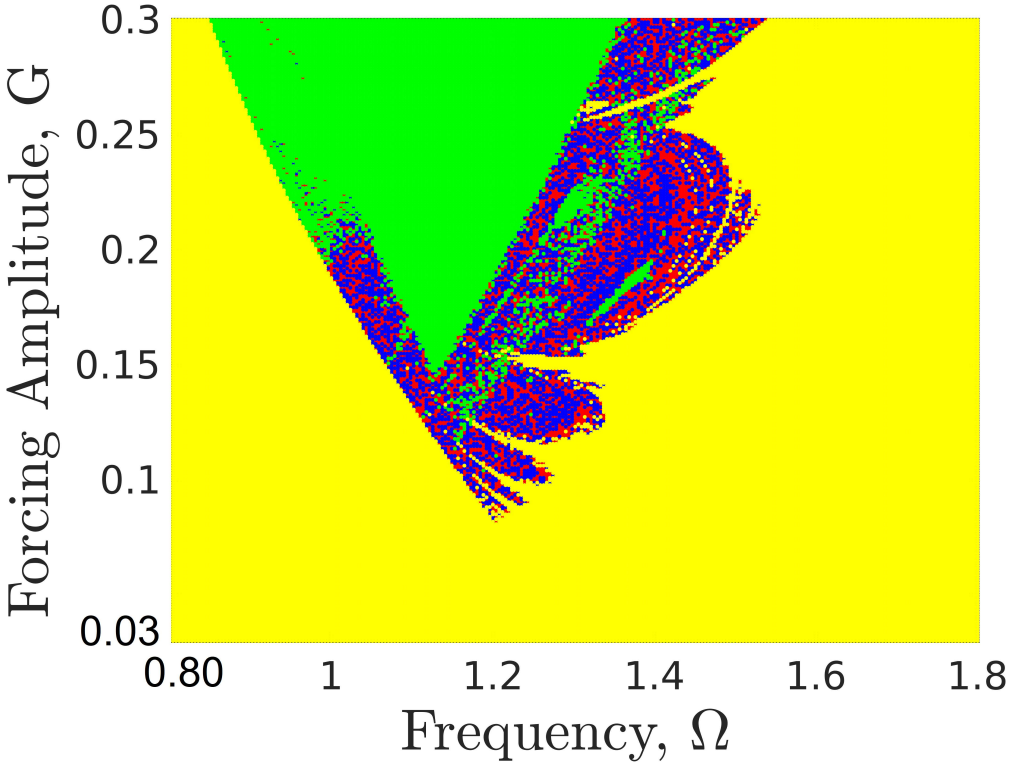


(e) $\gamma = 0.04$

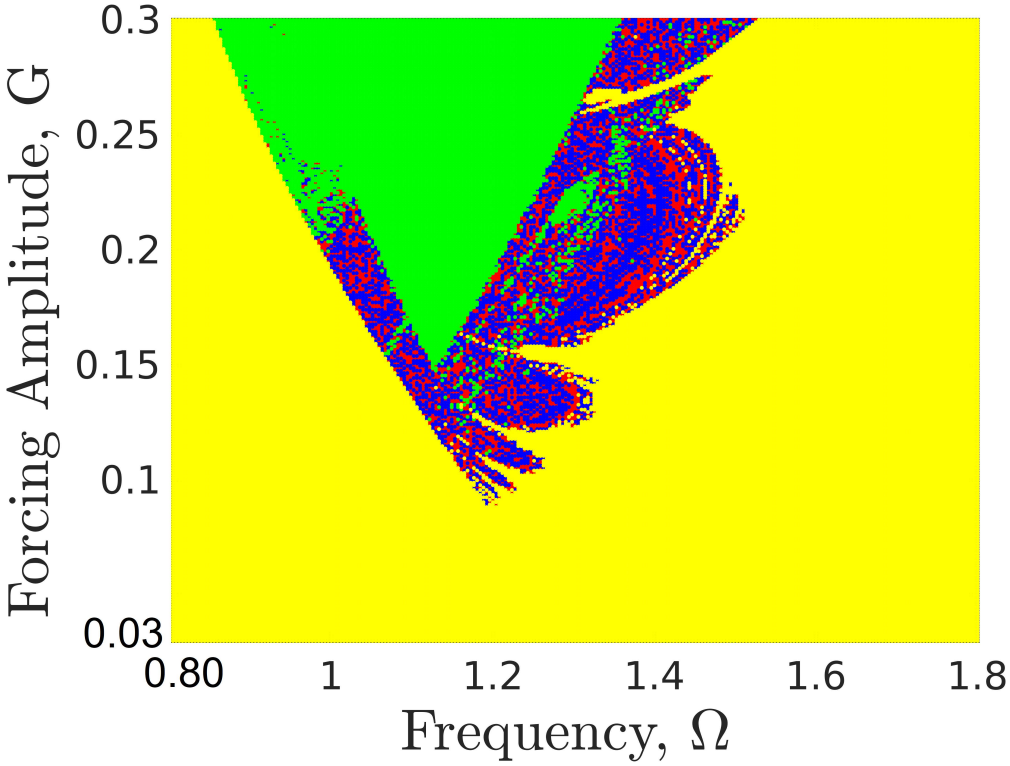


(f) $\gamma = 0.05$

Figure S6: Forcing amplitude-frequency parameter space

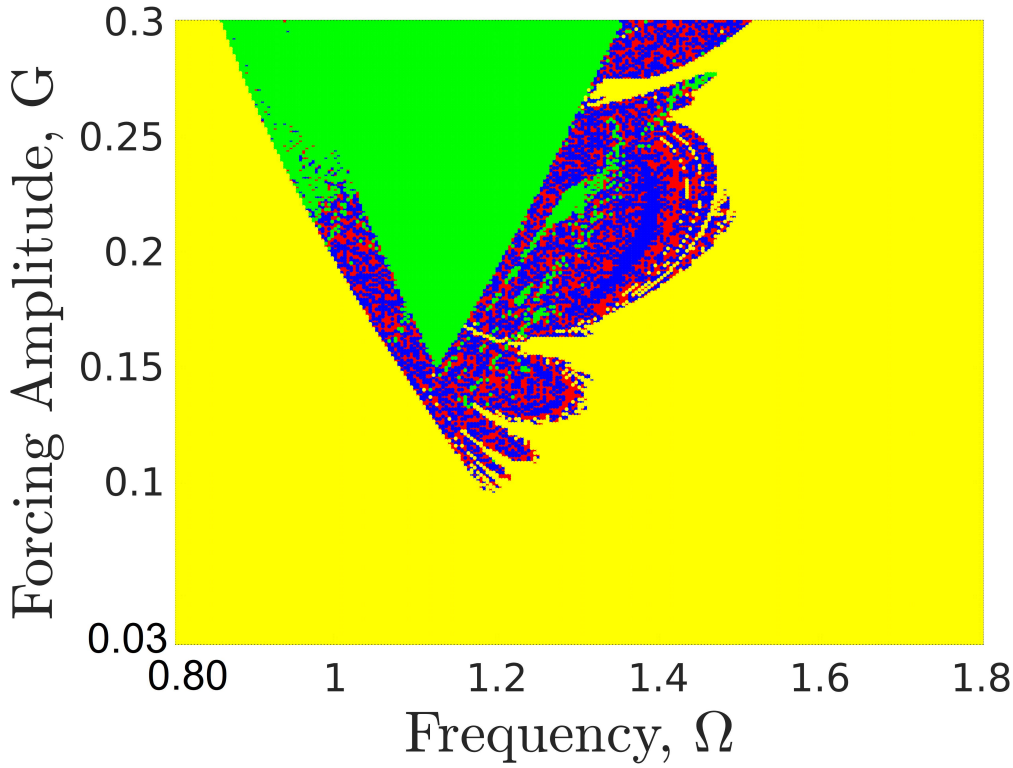


(g) $\gamma = 0.06$

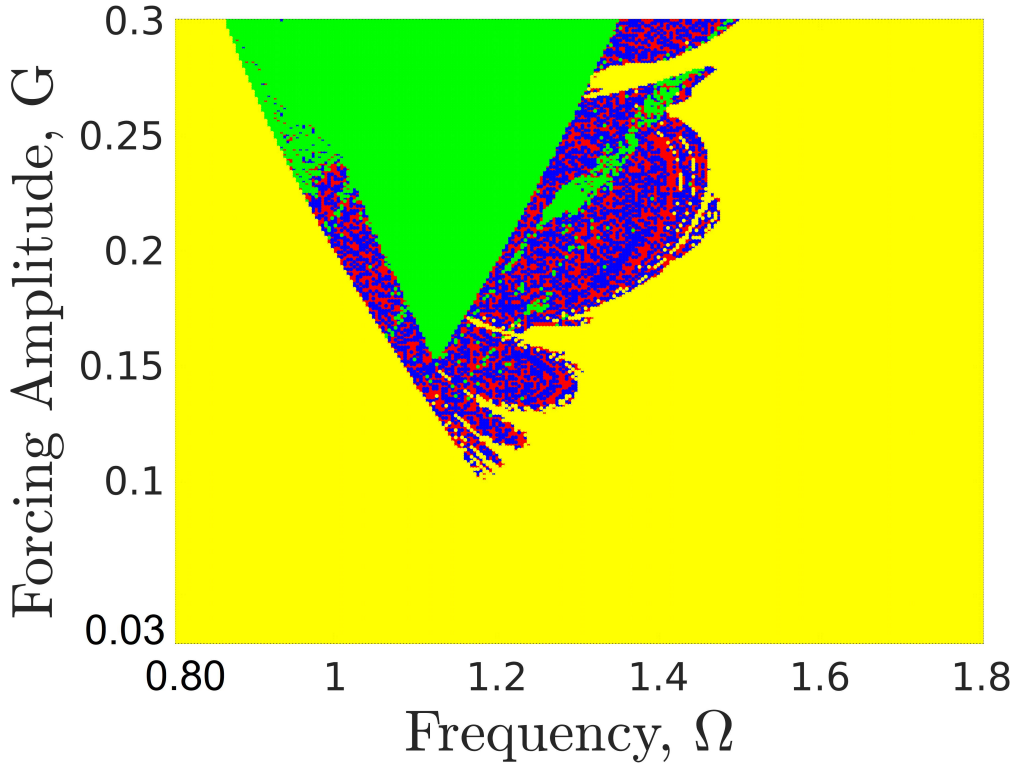


(h) $\gamma = 0.07$

Figure S6: Forcing amplitude-frequency parameter space

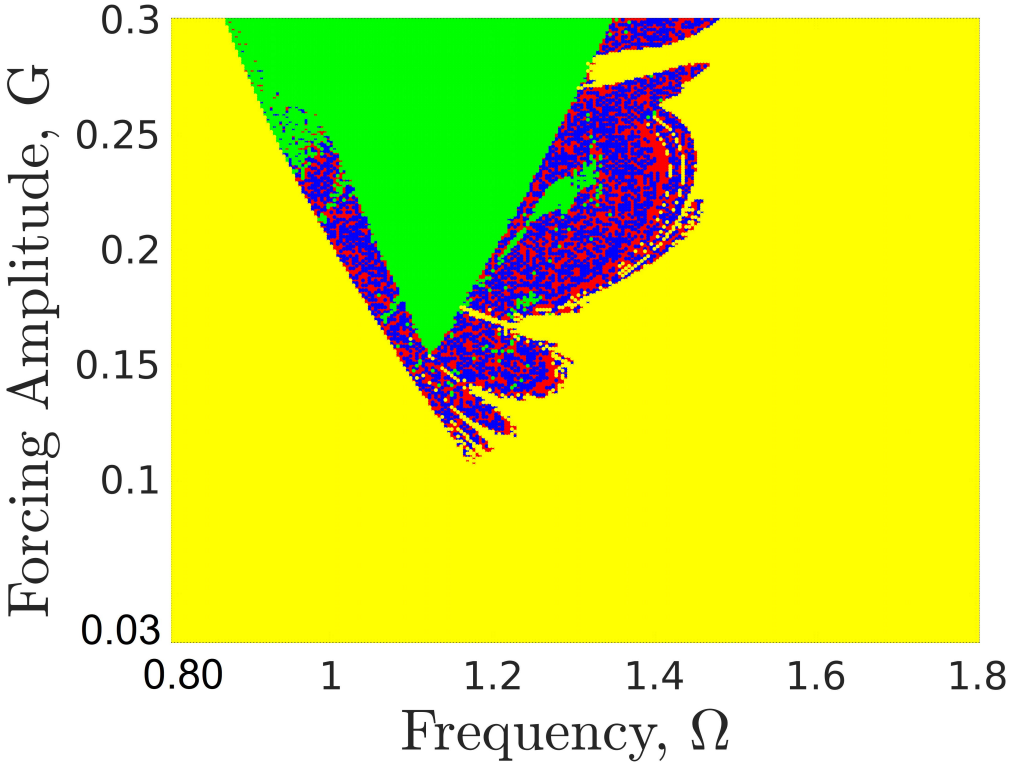


(i) $\gamma = 0.08$

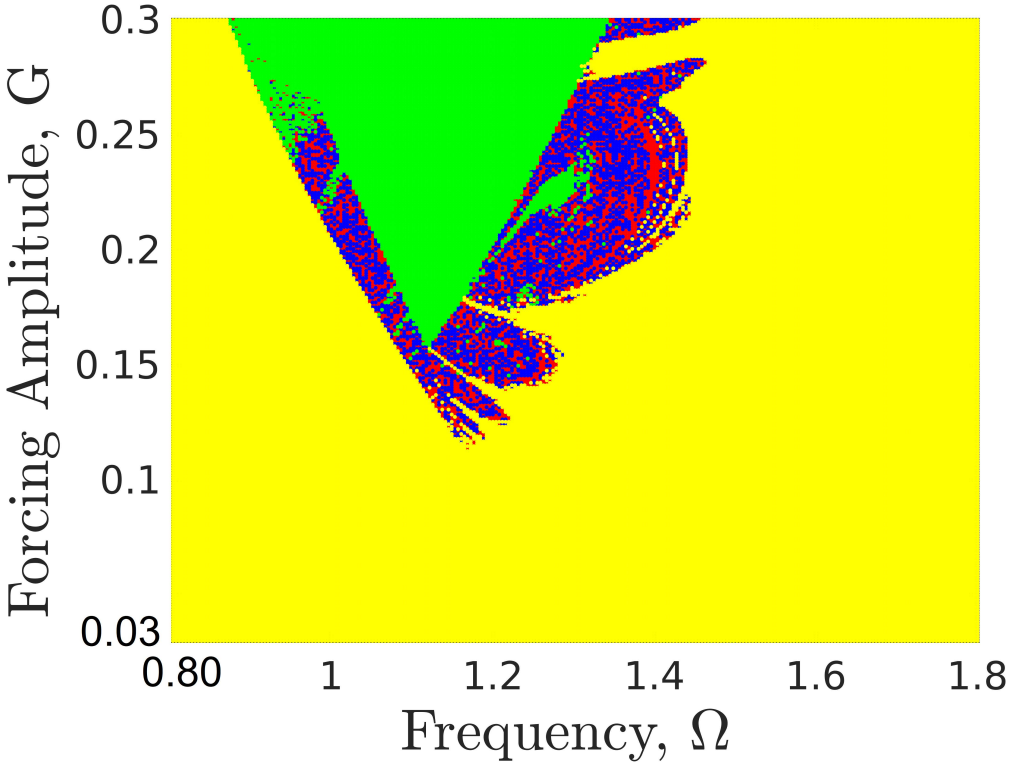


(j) $\gamma = 0.09$

Figure S6: Forcing amplitude-frequency parameter space

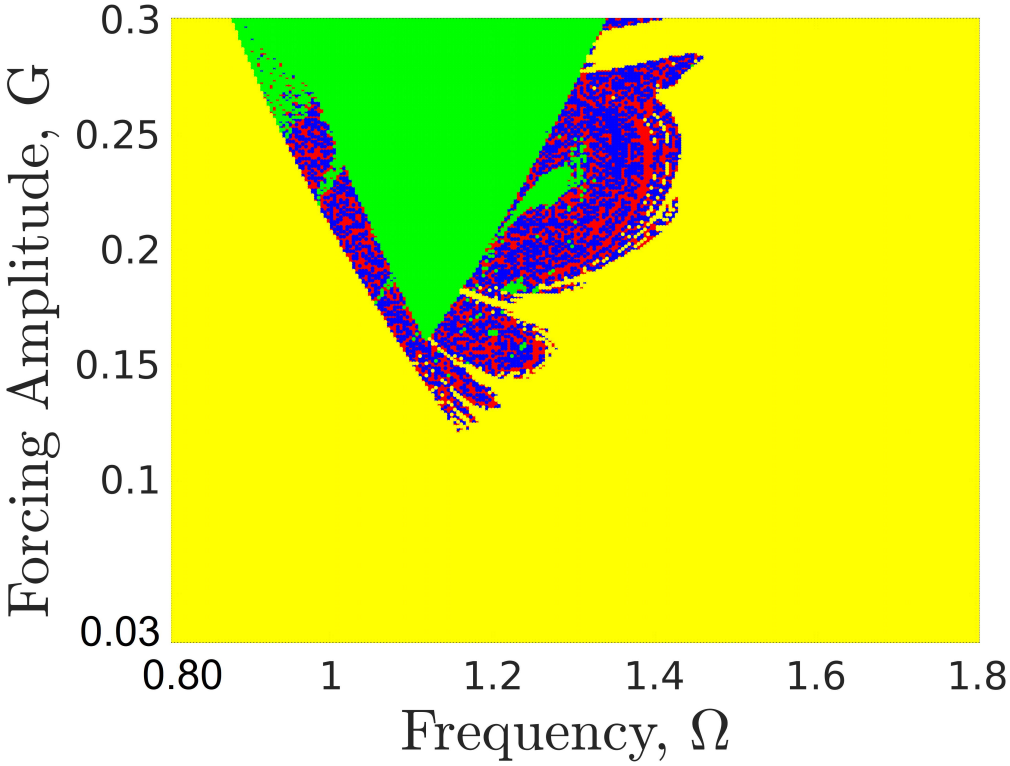


(k) $\gamma = 0.10$

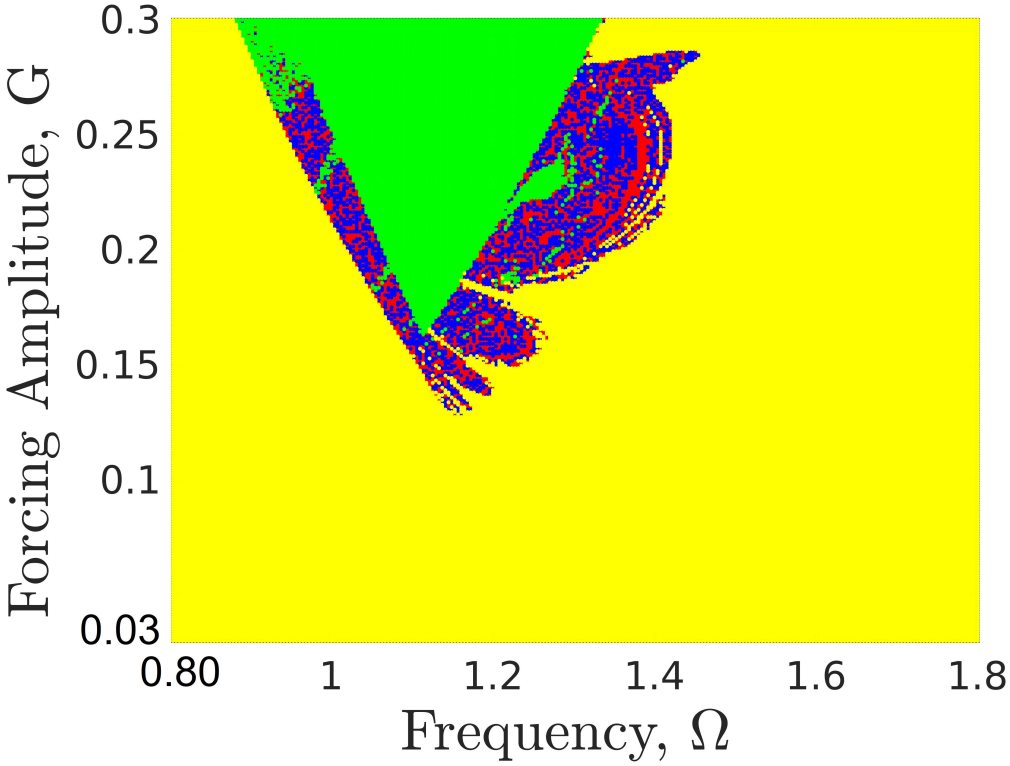


(l) $\gamma = 0.11$

Figure S6: Forcing amplitude-frequency parameter space

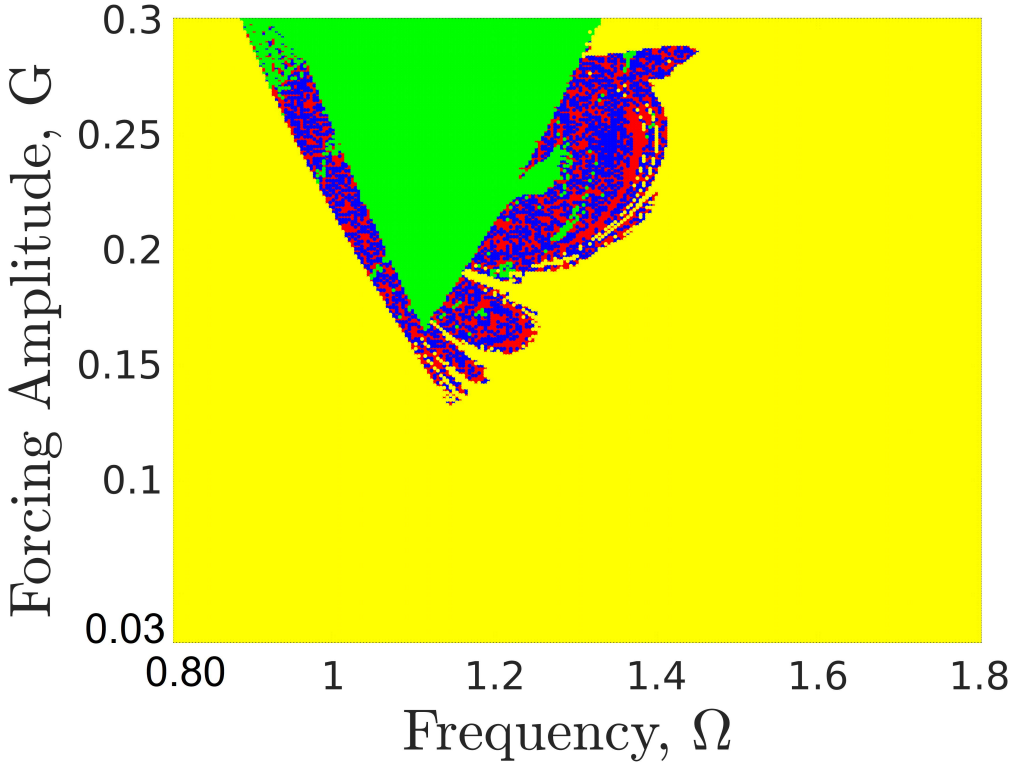


(m) $\gamma = 0.12$

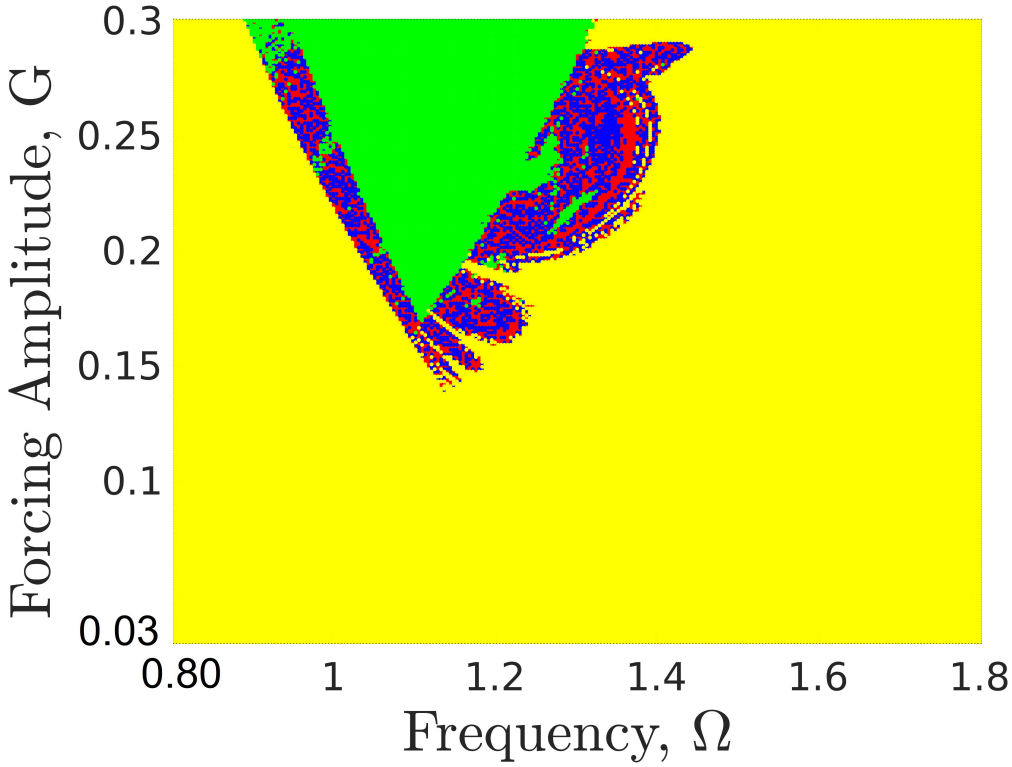


(n) $\gamma = 0.13$

Figure S6: Forcing amplitude-frequency parameter space

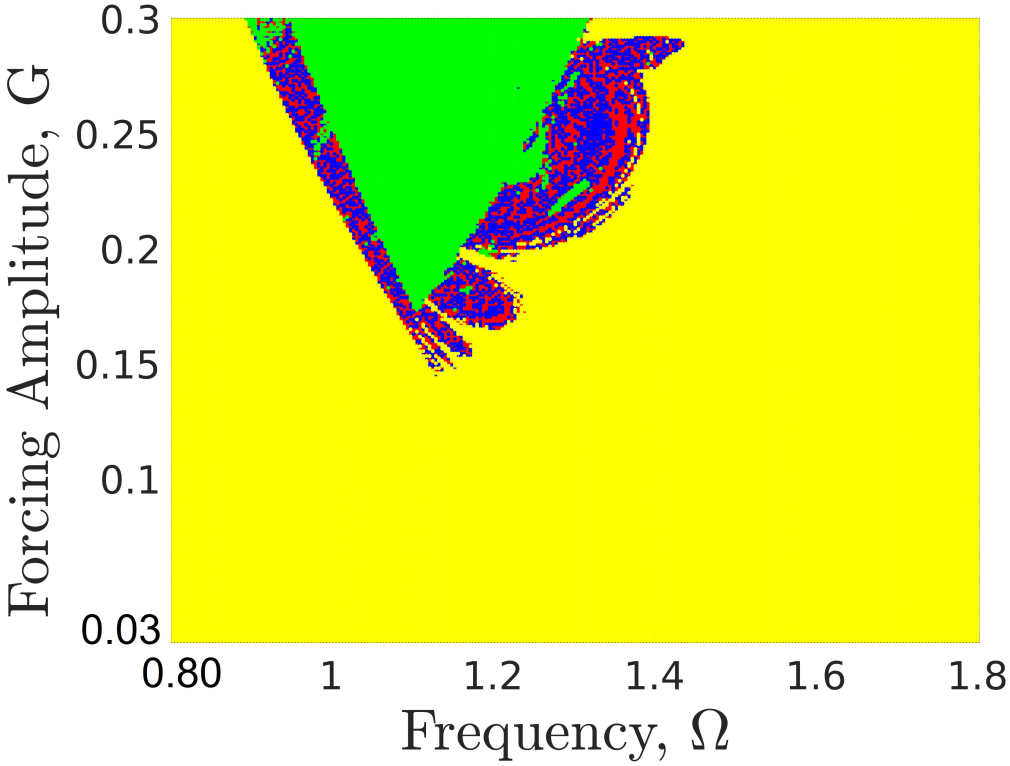


(o) $\gamma = 0.14$

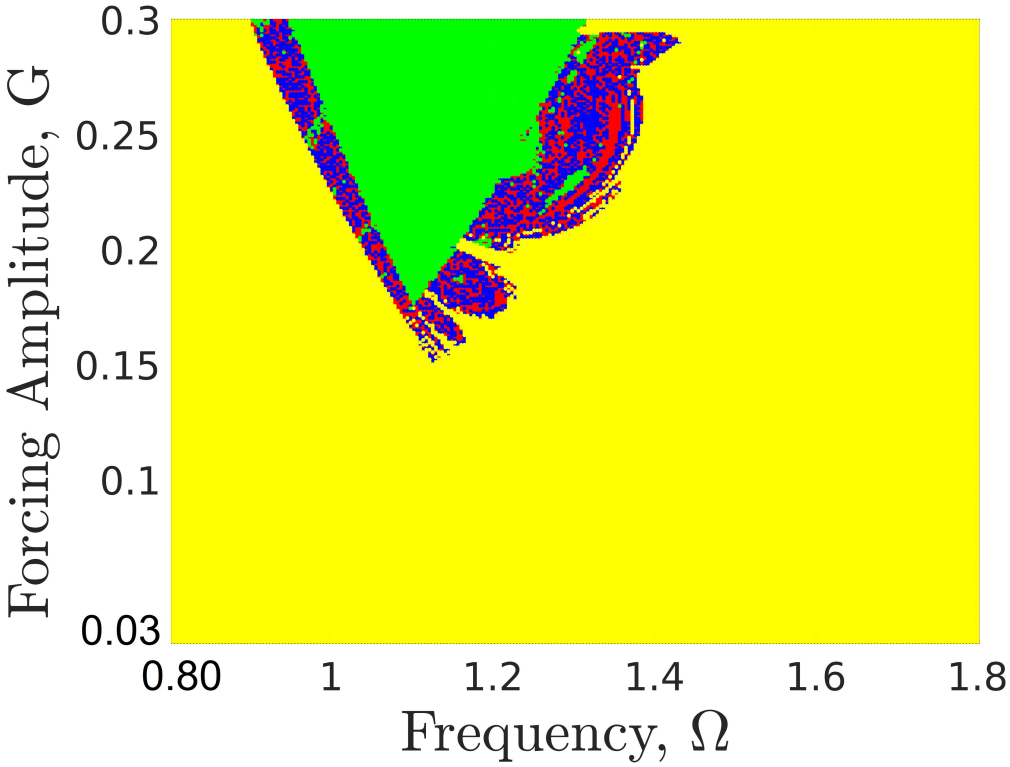


(p) $\gamma = 0.15$

Figure S6: Forcing amplitude-frequency parameter space

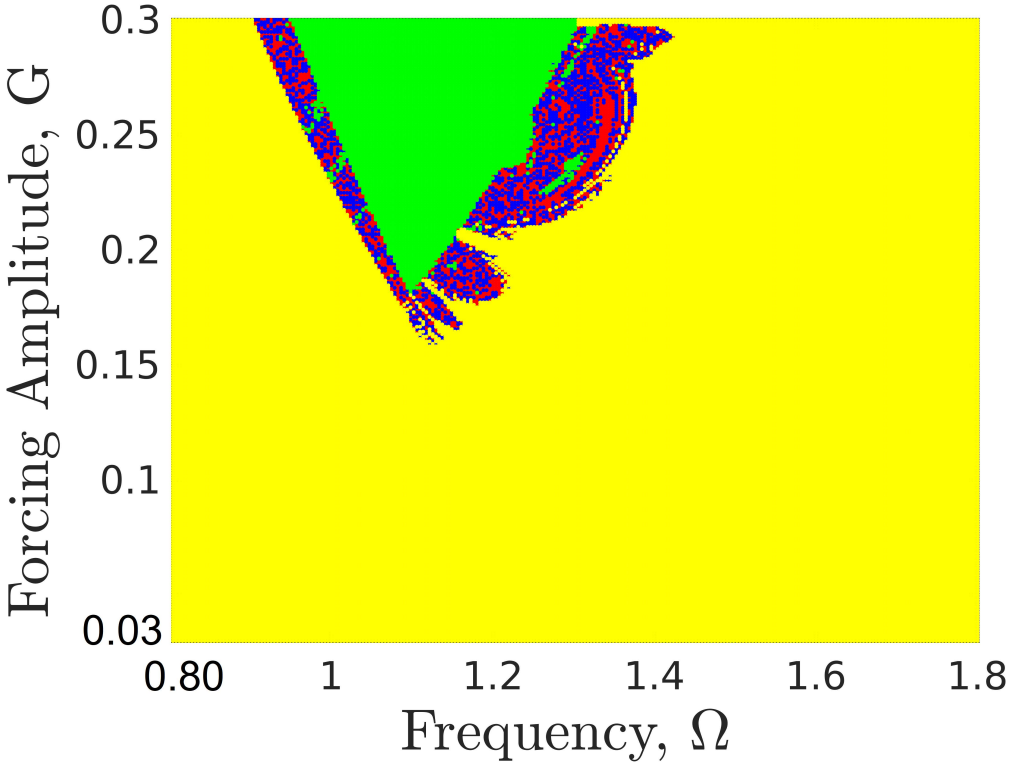


(q) $\gamma = 0.16$

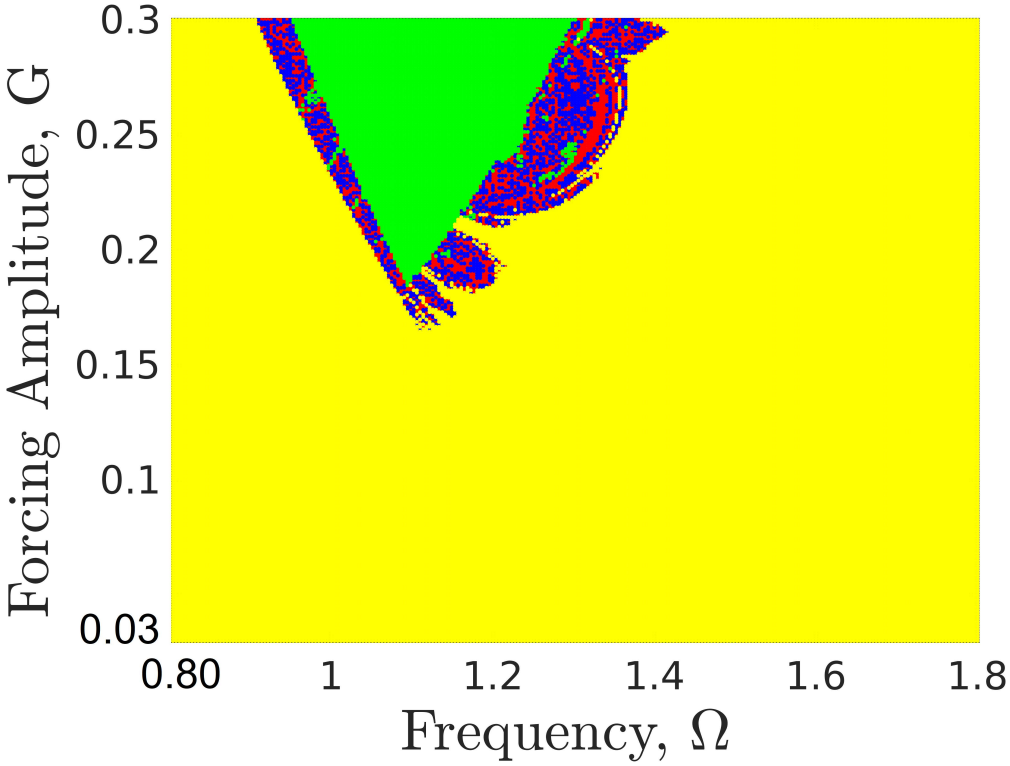


(r) $\gamma = 0.17$

Figure S6: Forcing amplitude-frequency parameter space

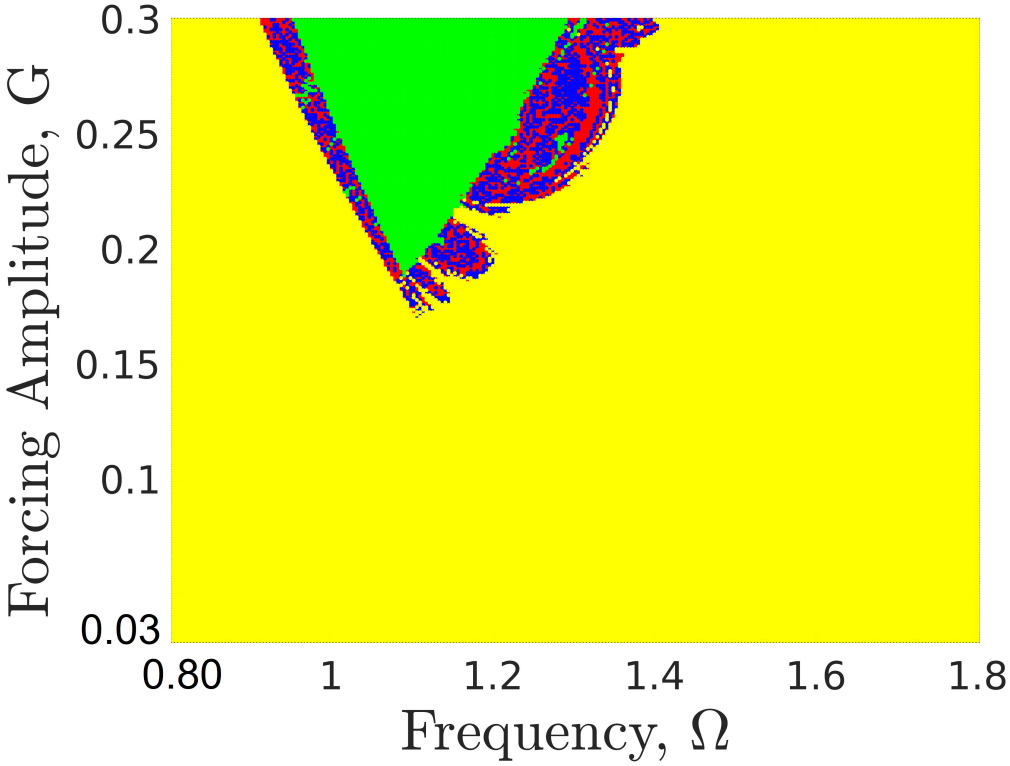


(s) $\gamma = 0.18$

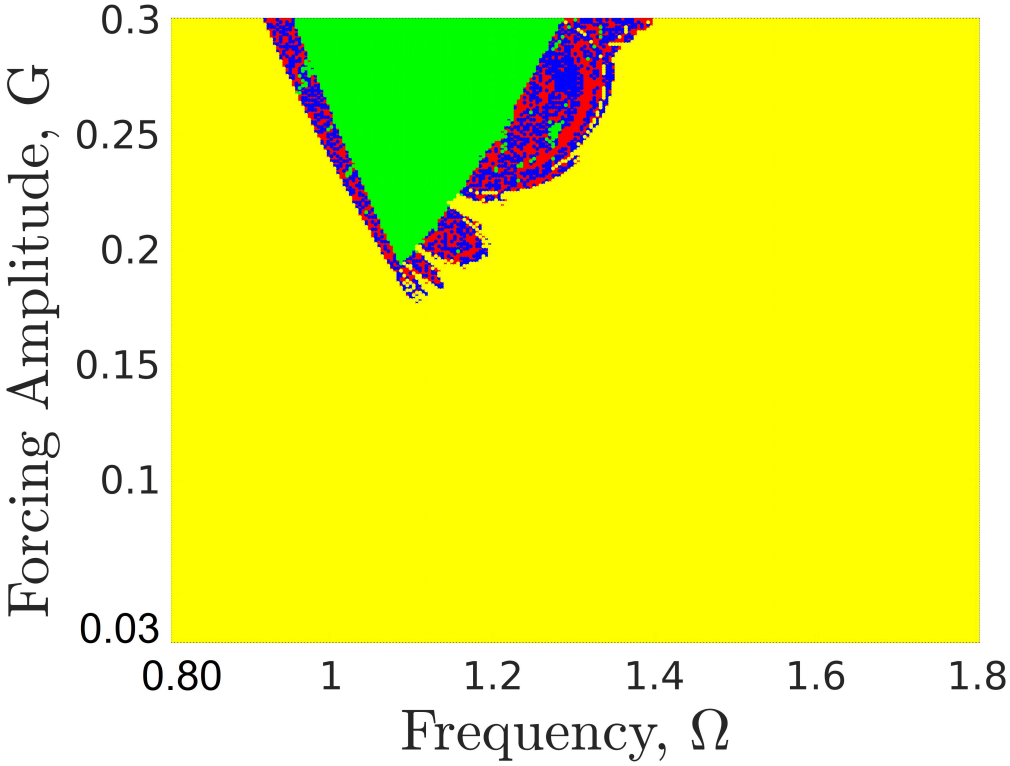


(t) $\gamma = 0.19$

Figure S6: Forcing amplitude-frequency parameter space

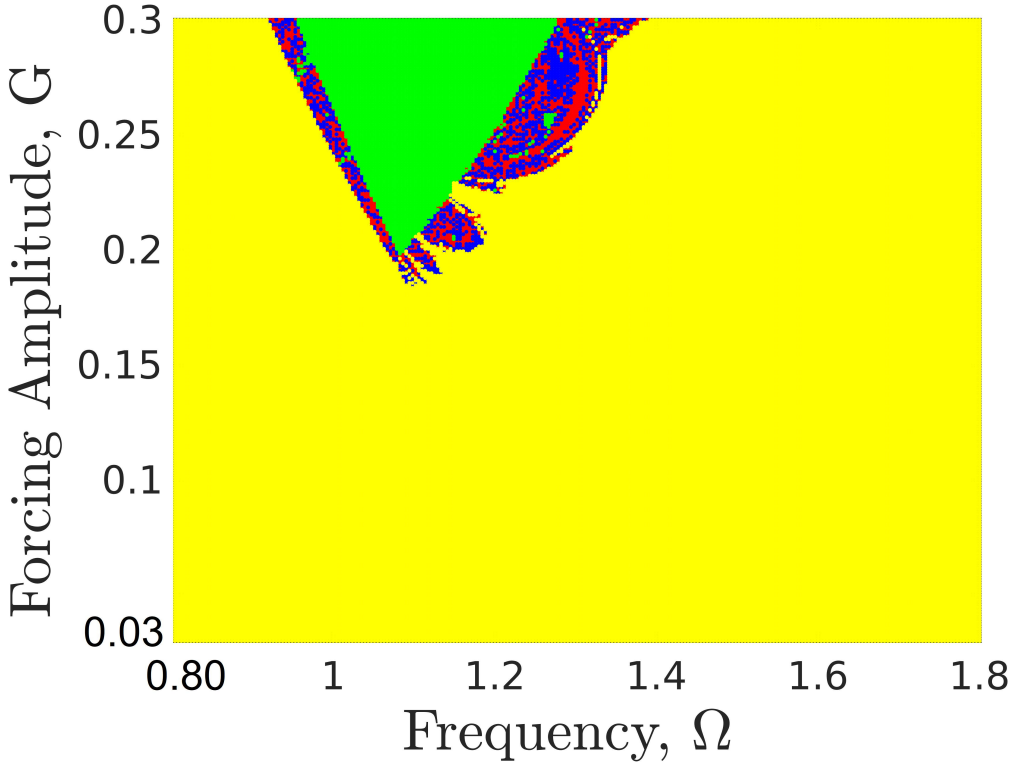


(u) $\gamma = 0.20$

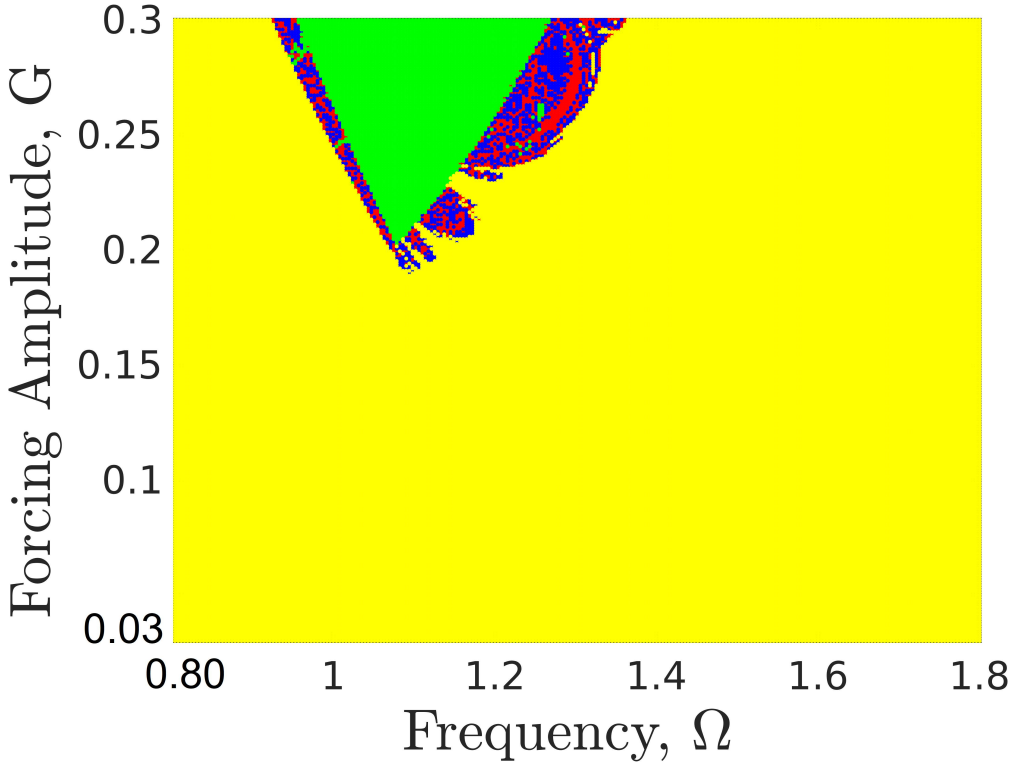


(v) $\gamma = 0.21$

Figure S6: Forcing amplitude-frequency parameter space

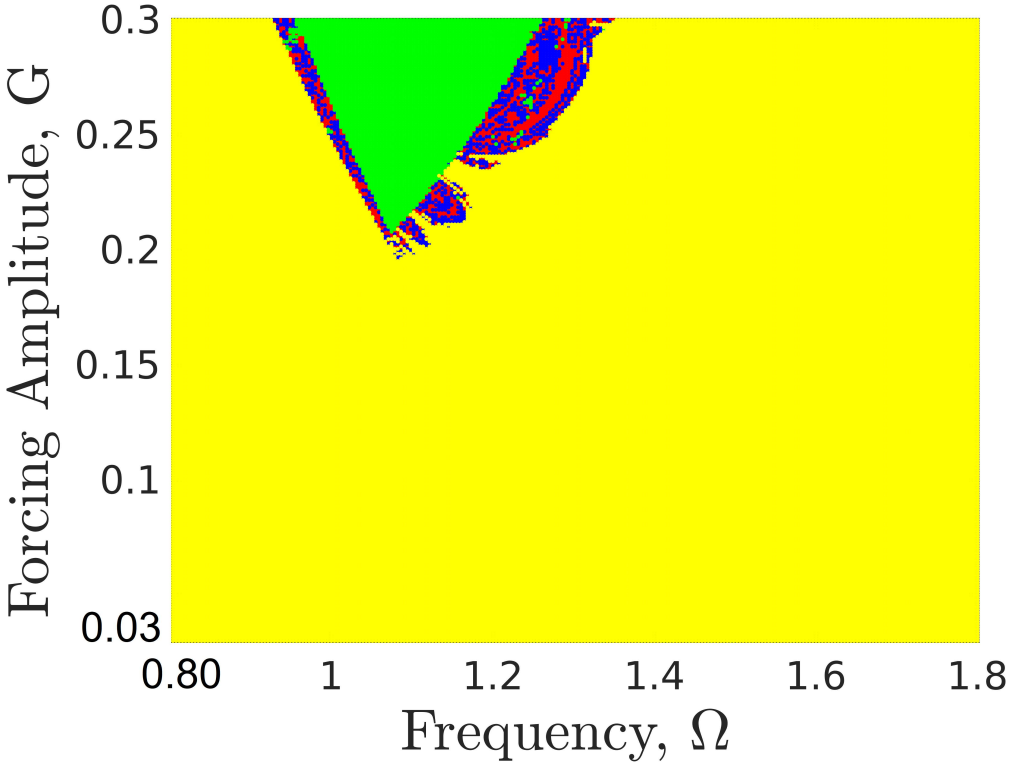


(w) $\gamma = 0.22$

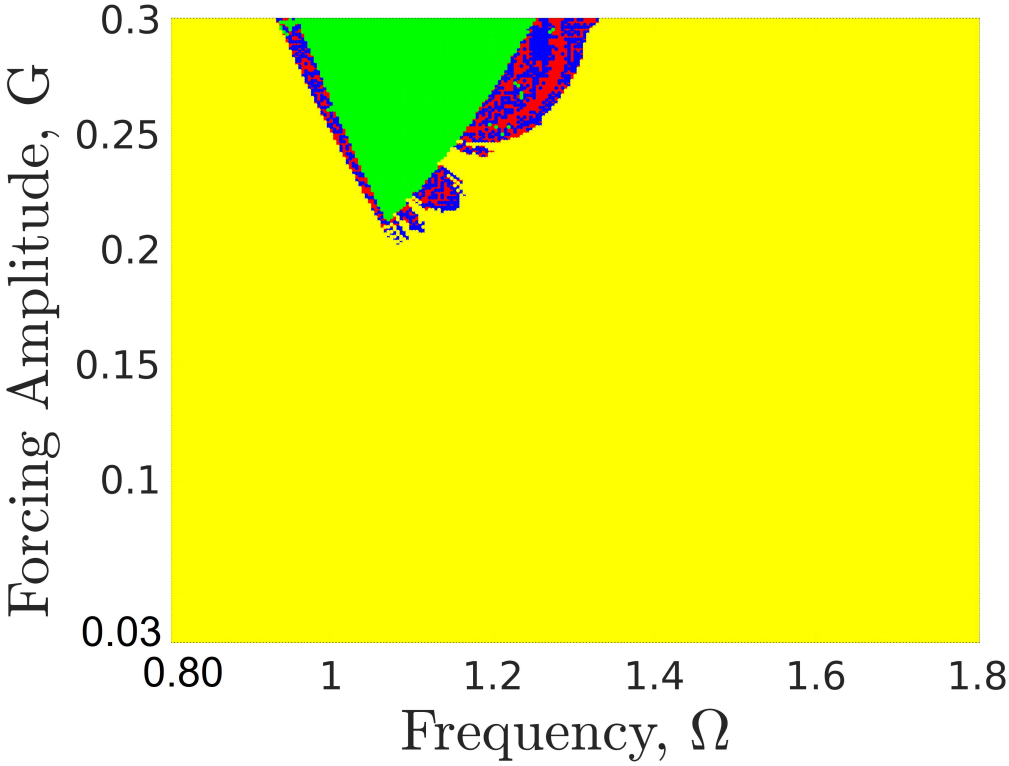


(x) $\gamma = 0.23$

Figure S6: Forcing amplitude-frequency parameter space

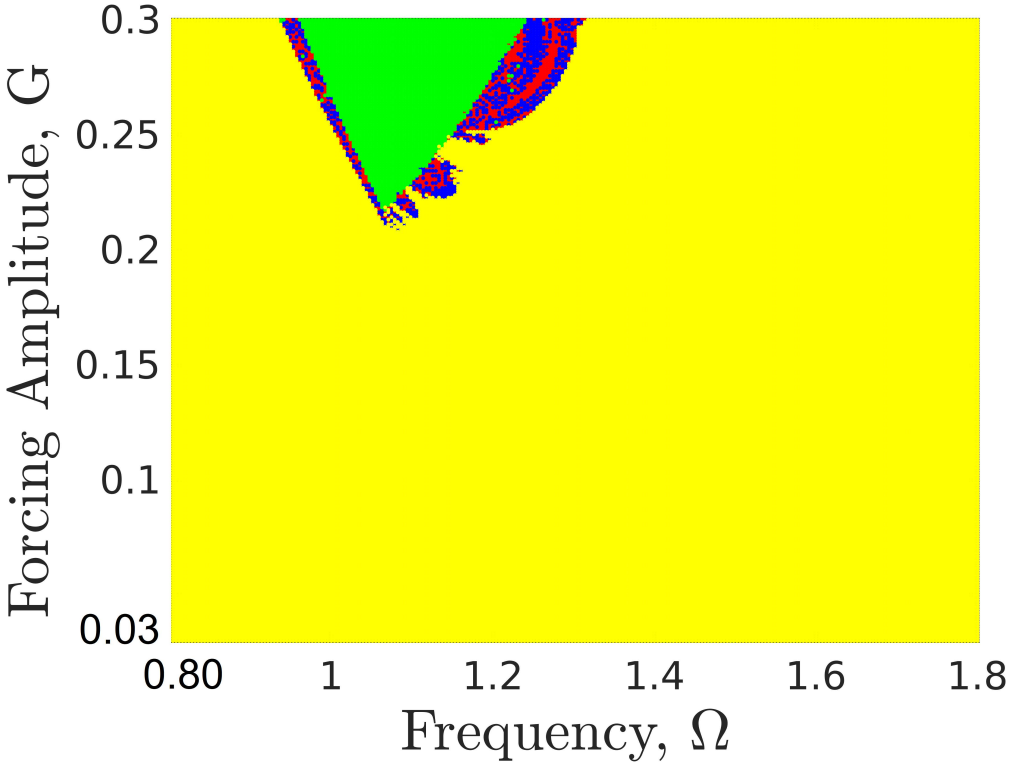


(y) $\gamma = 0.24$

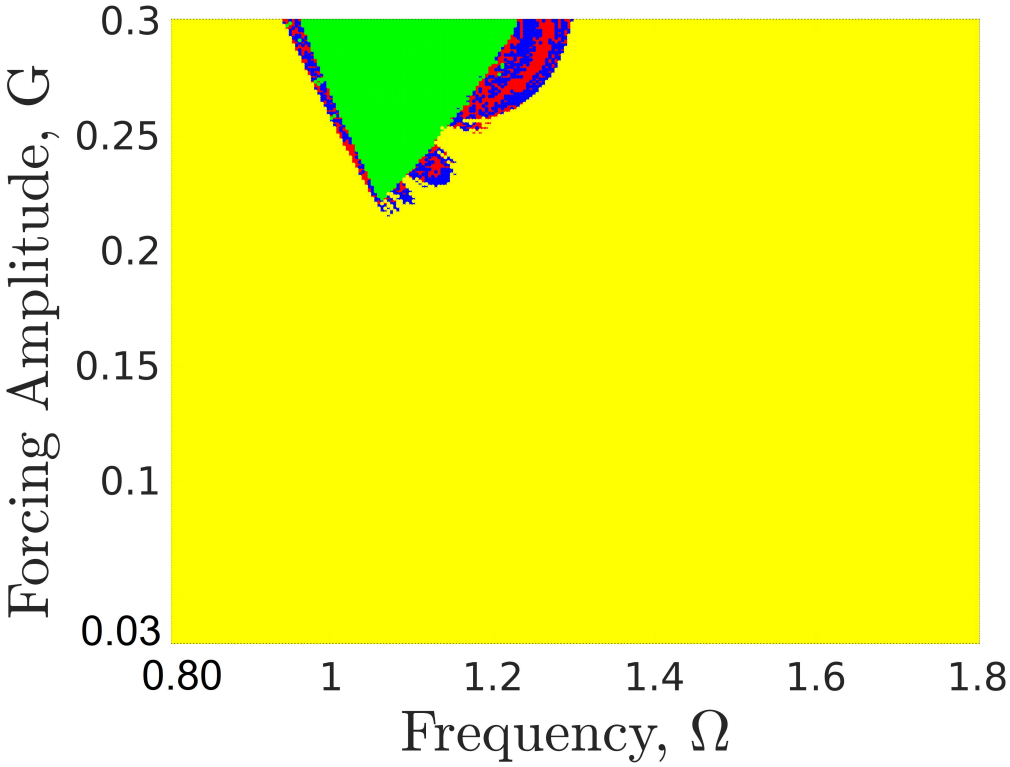


(z) $\gamma = 0.25$

Figure S6: Forcing amplitude-frequency parameter space

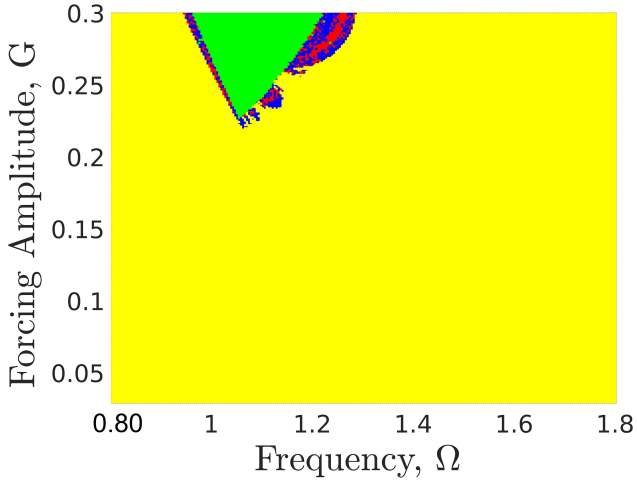


(aa) $\gamma = 0.26$

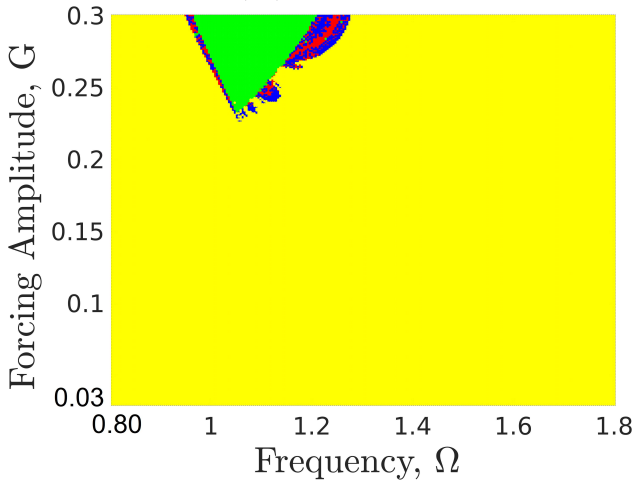


(ab) $\gamma = 0.27$

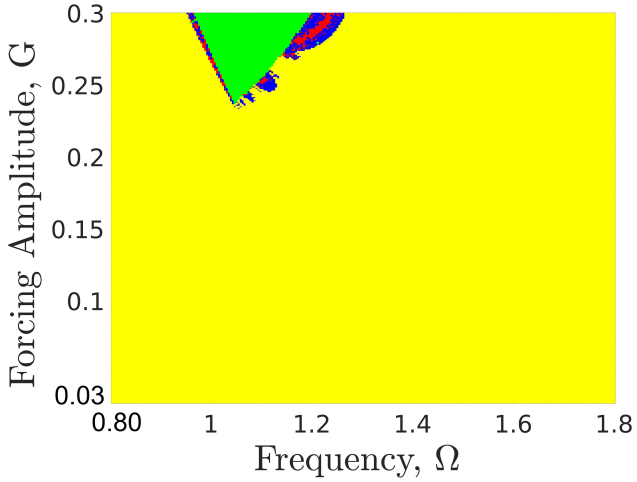
Figure S6: Forcing amplitude-frequency parameter space



(ac) $\gamma = 0.28$



(ad) $\gamma = 0.29$

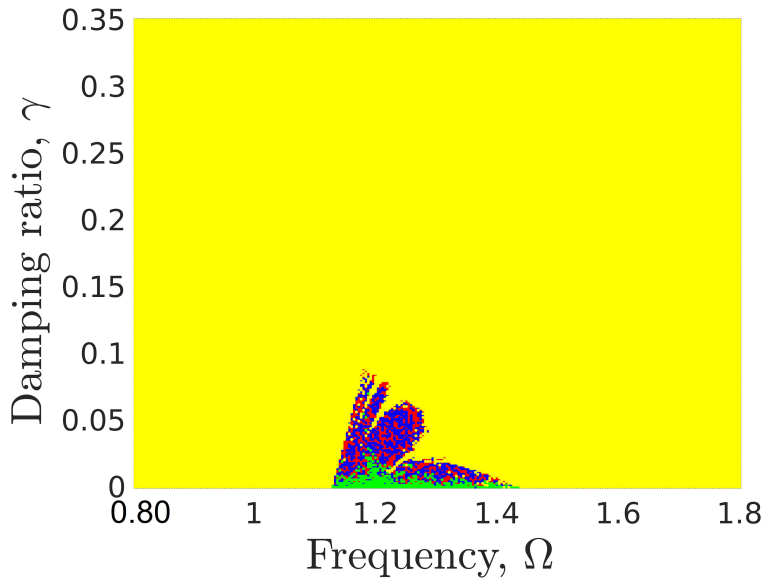


(ae) $\gamma = 0.30$

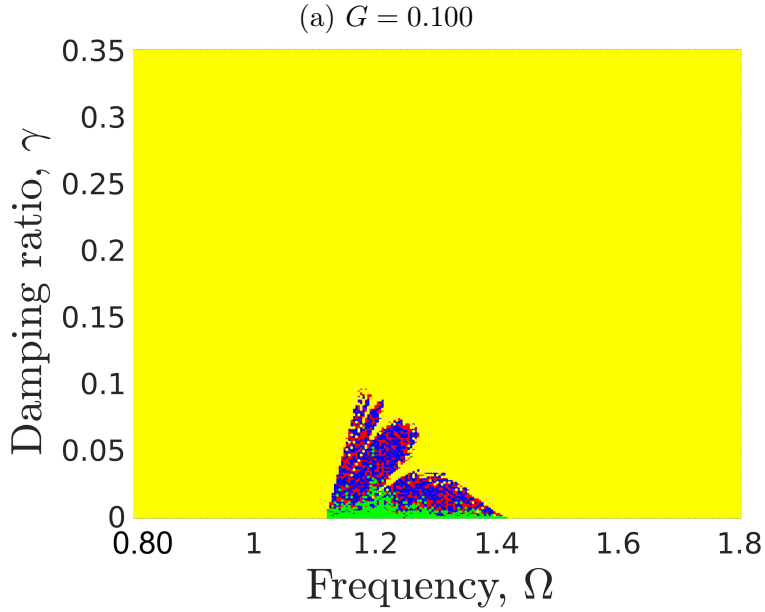
Figure S6: Forcing amplitude-frequency parameter space

S5 Damping Ratio-Forcing Frequency (γ vs. Ω) Parameter Space

31 damping ratio-forcing frequency (γ vs. Ω) parameter space for G ranges from $0.100 \sim 0.250$ with the increment of 0.05.

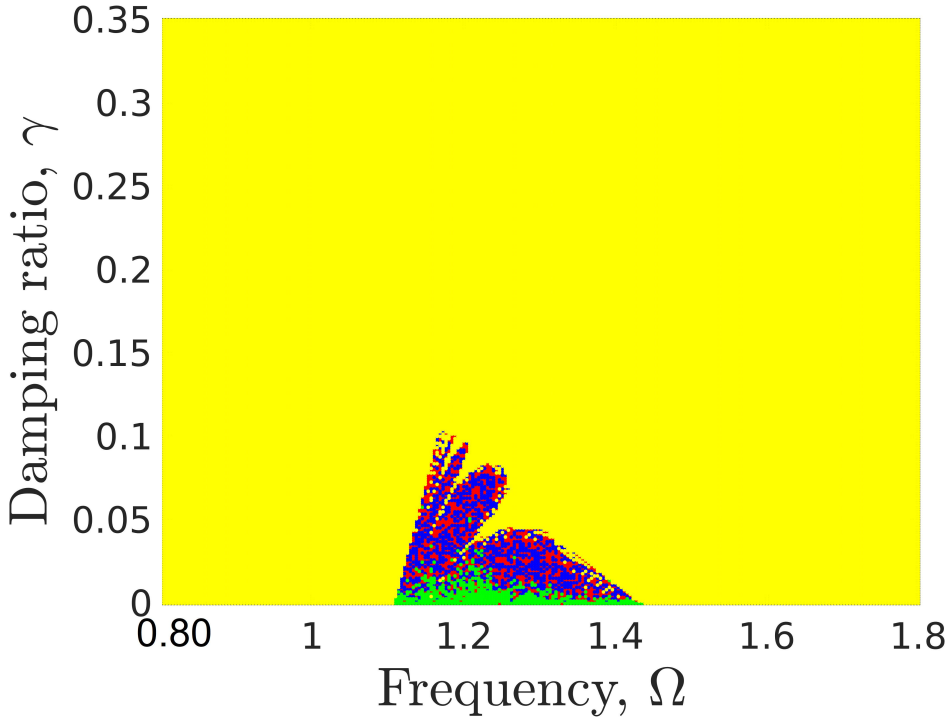


(a) $G = 0.100$

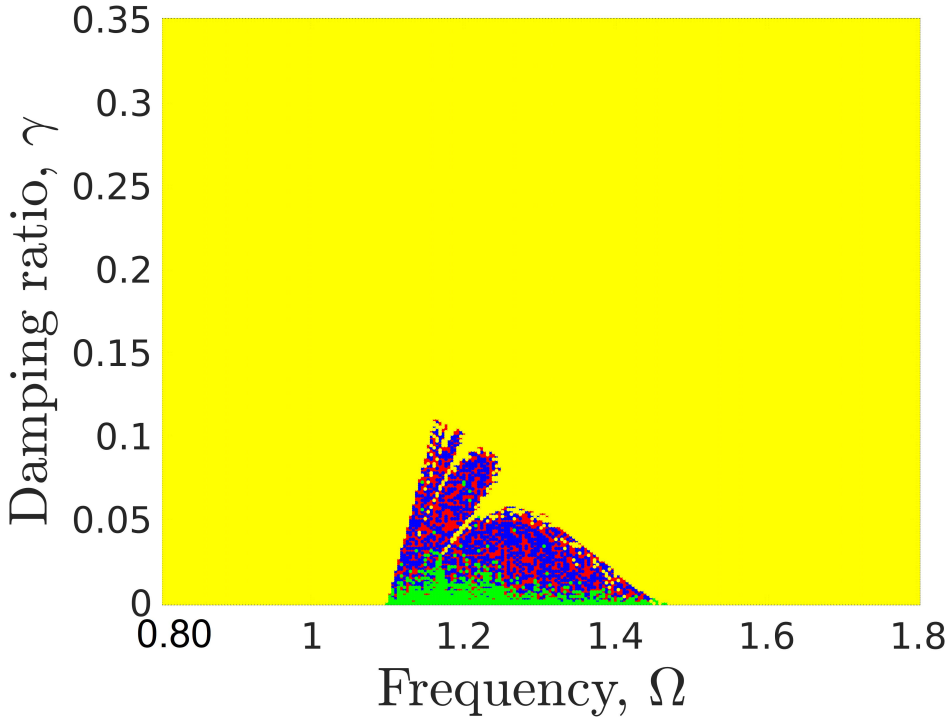


(b) $G = 0.105$

Figure S7: Damping ratio-forcing frequency parameter space

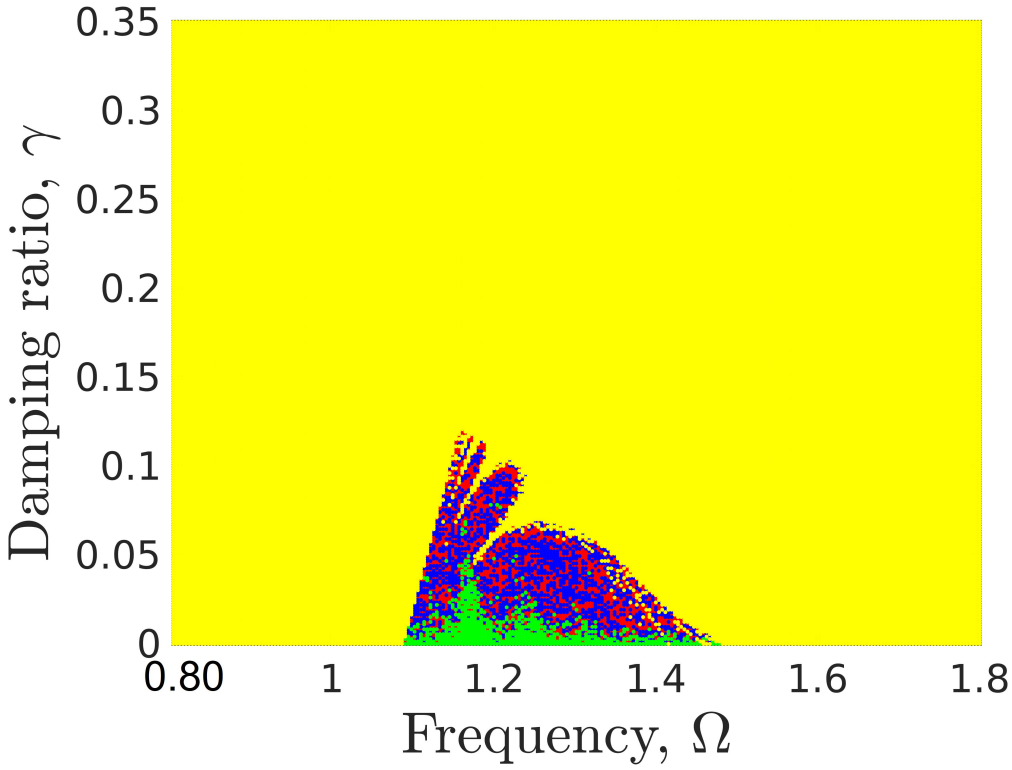


(c) $G = 0.110$

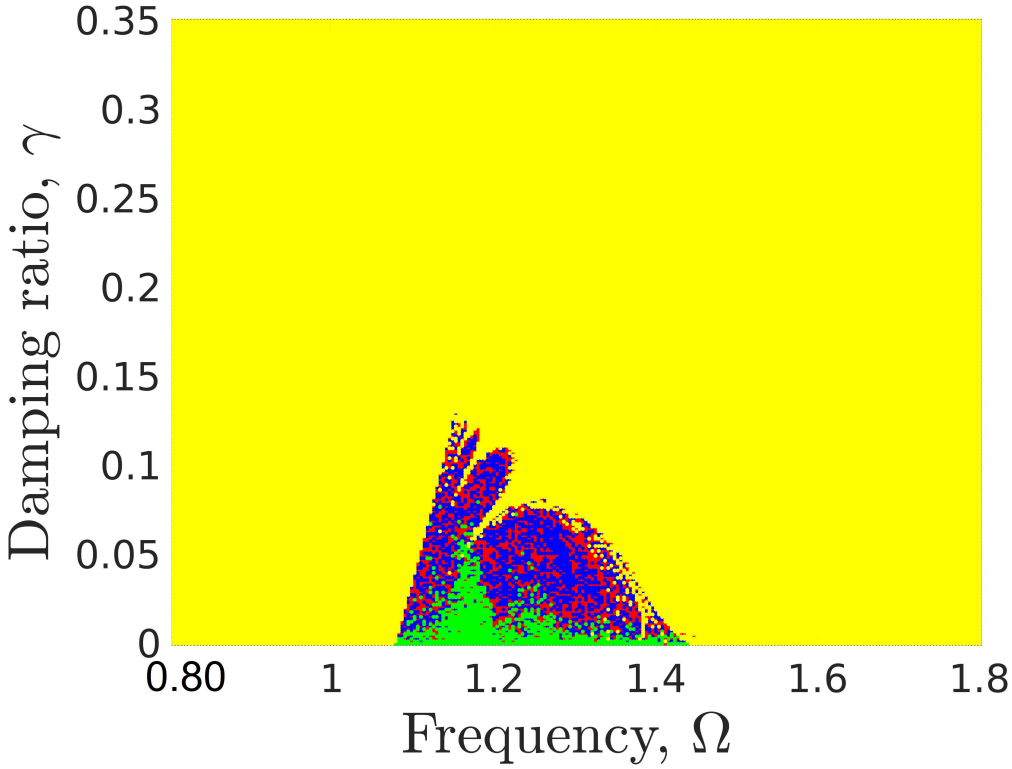


(d) $G = 0.115$

Figure S7: Damping ratio-forcing frequency parameter space

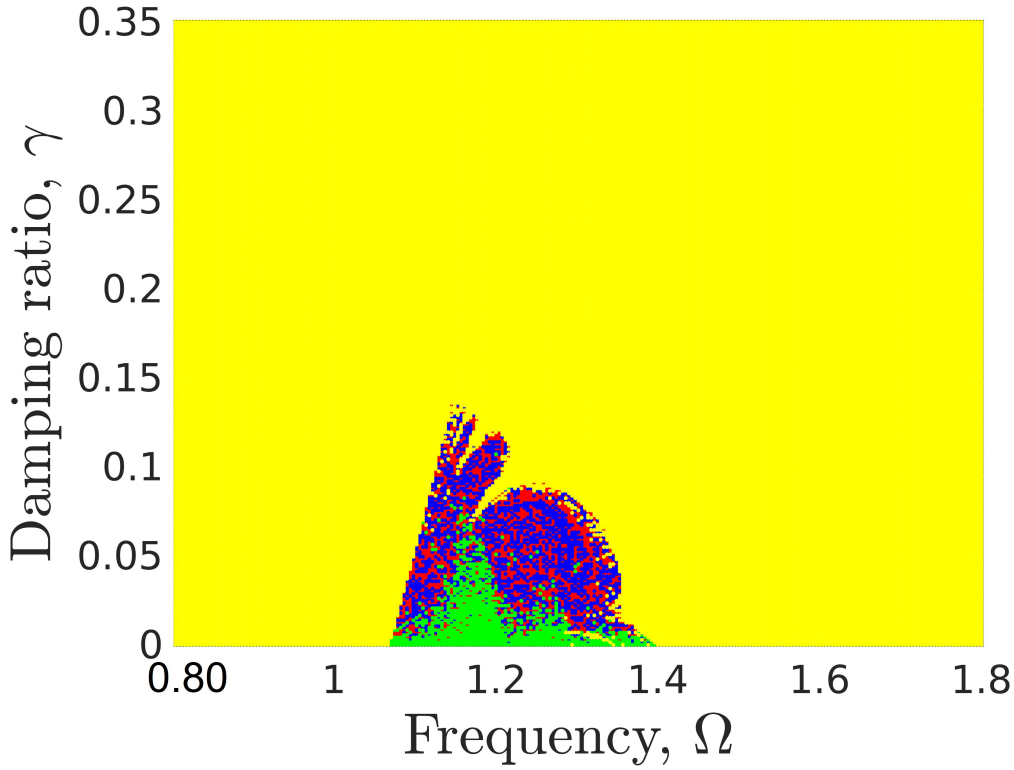


(e) $G = 0.120$

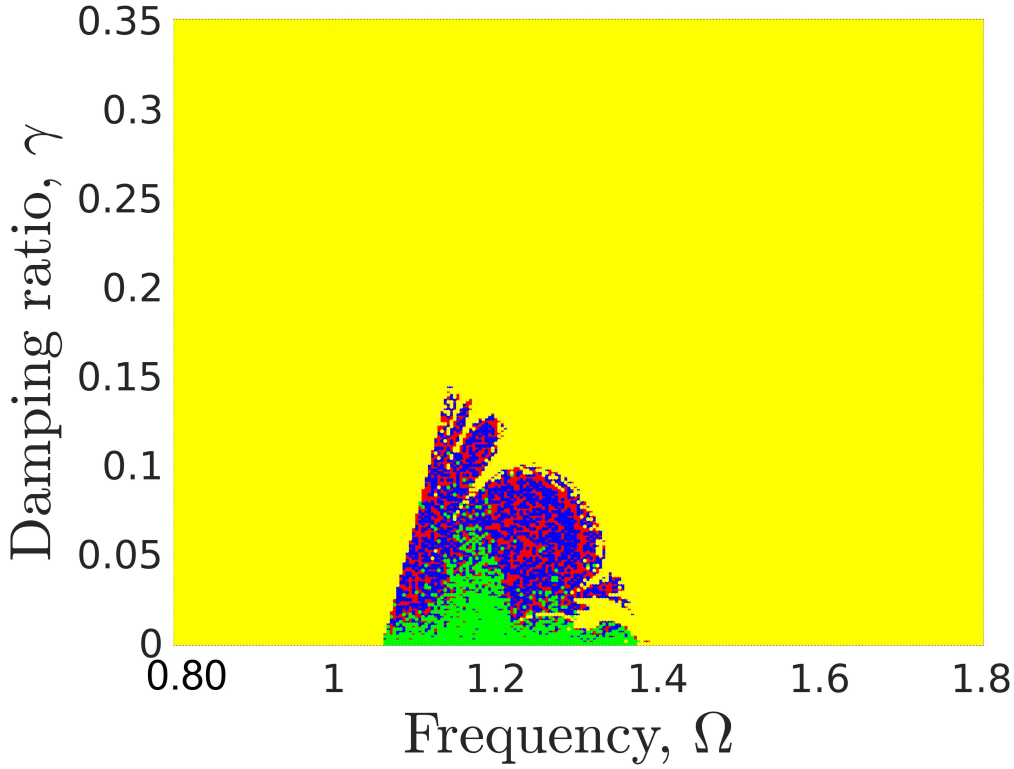


(f) $G = 0.125$

Figure S7: Damping ratio-forcing frequency parameter space

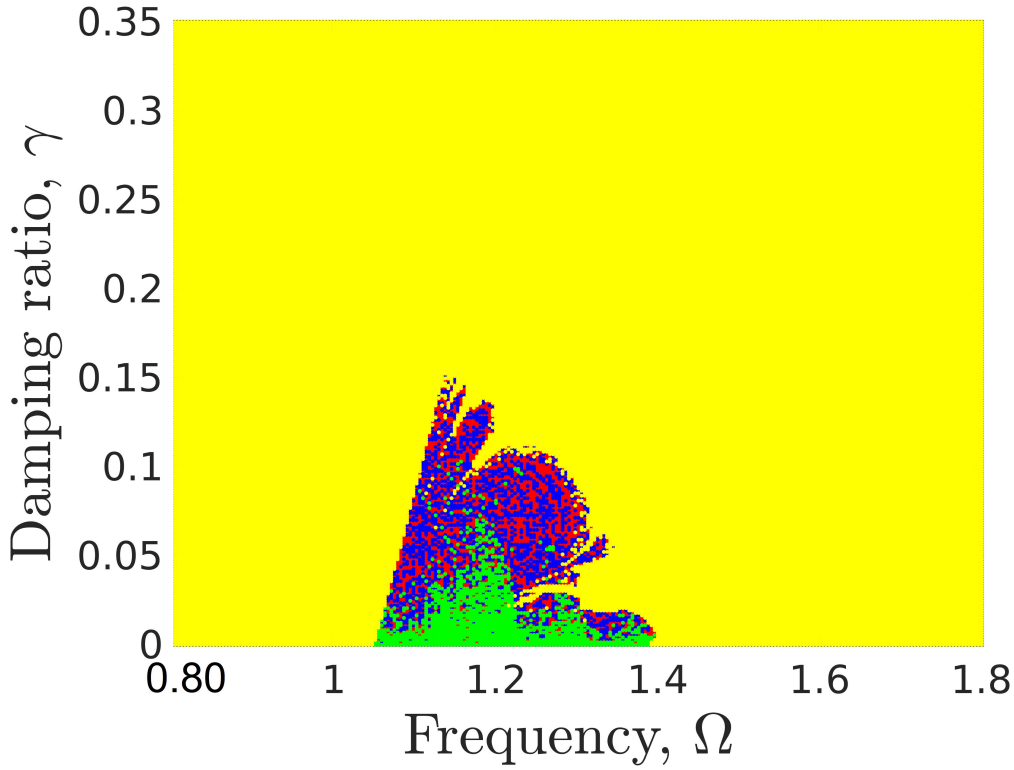


(g) $G = 0.130$

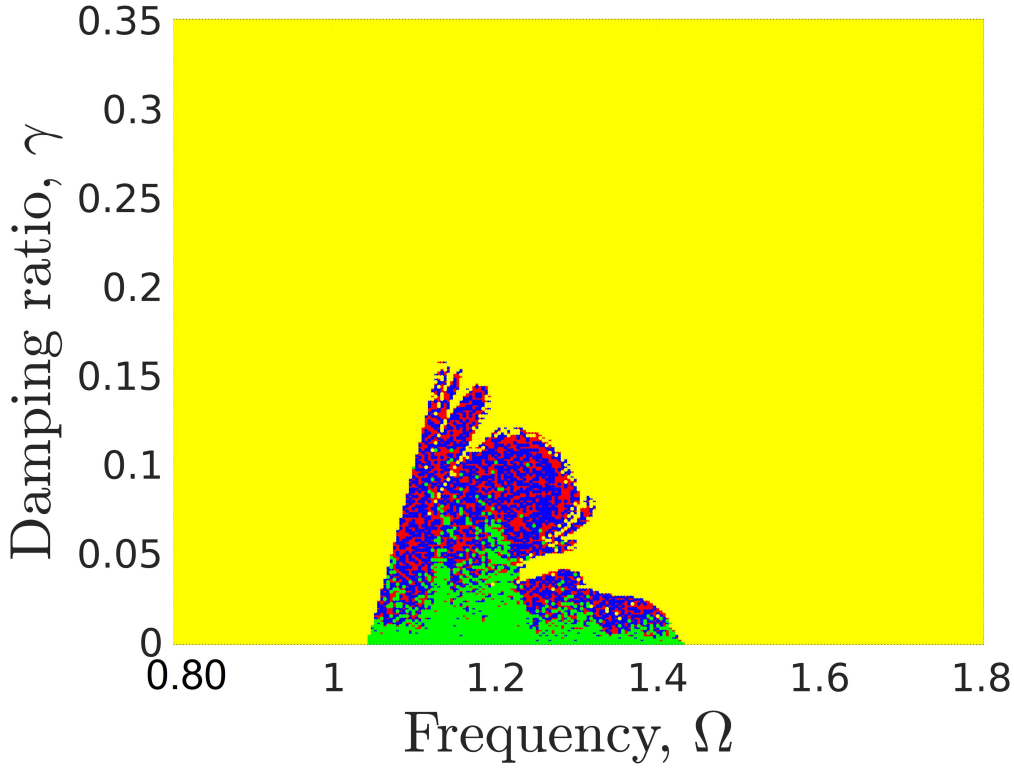


(h) $G = 0.135$

Figure S7: Damping ratio-forcing frequency parameter space

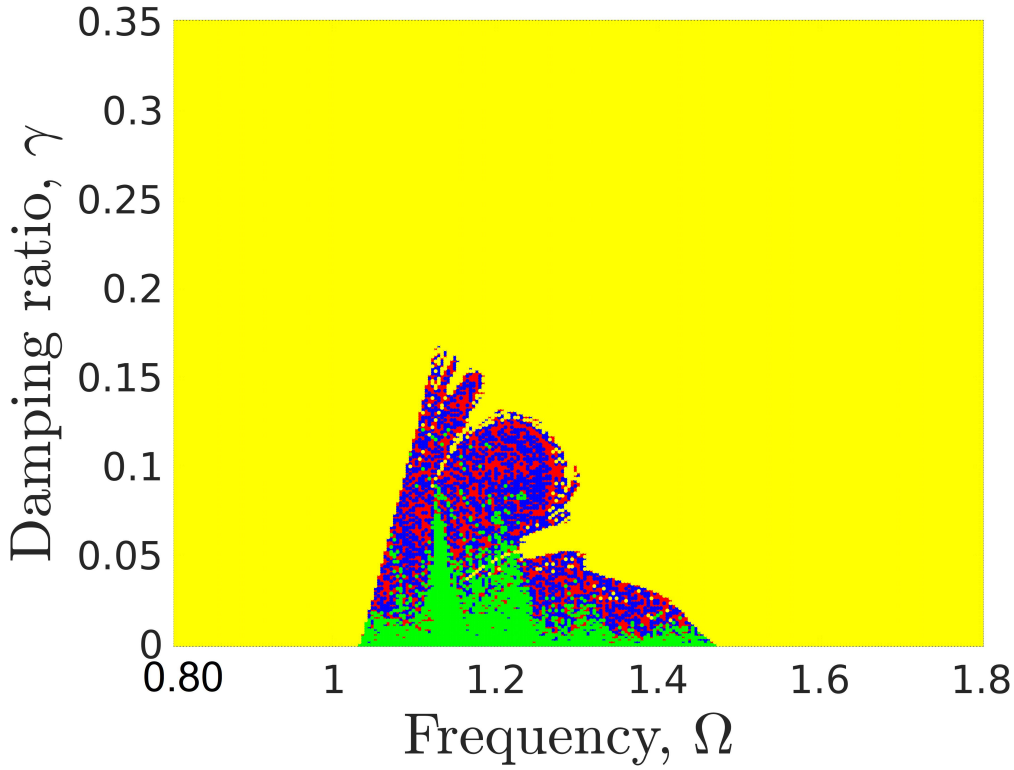


(i) $G = 0.140$

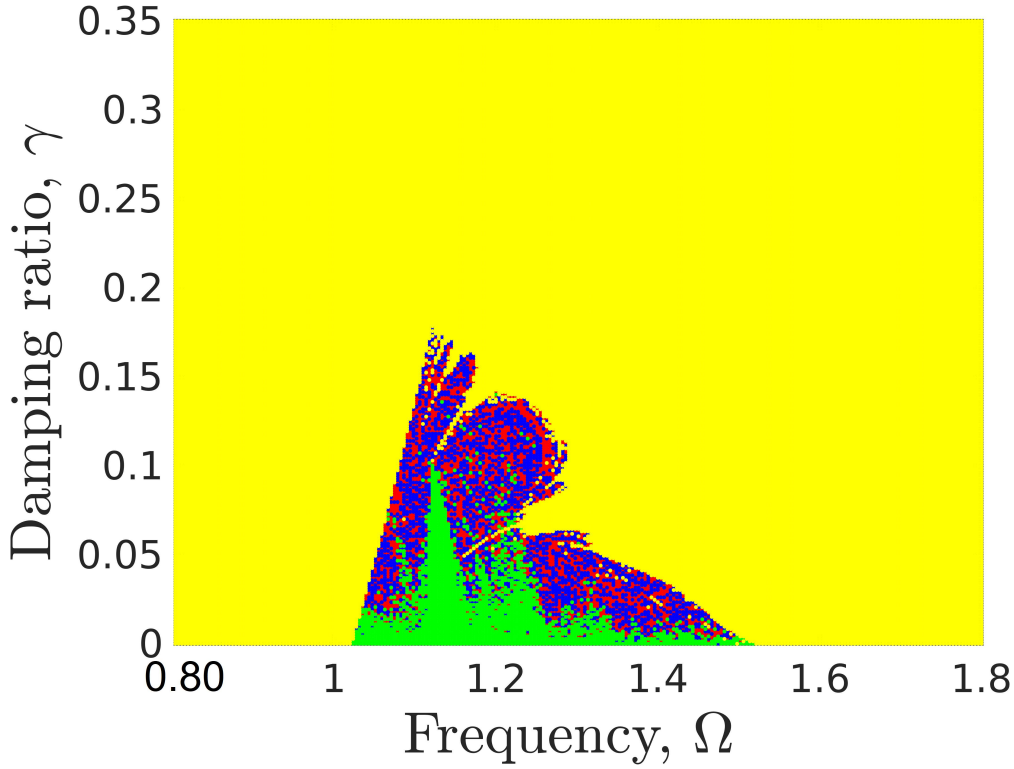


(j) $G = 0.145$

Figure S7: Damping ratio-forcing frequency parameter space

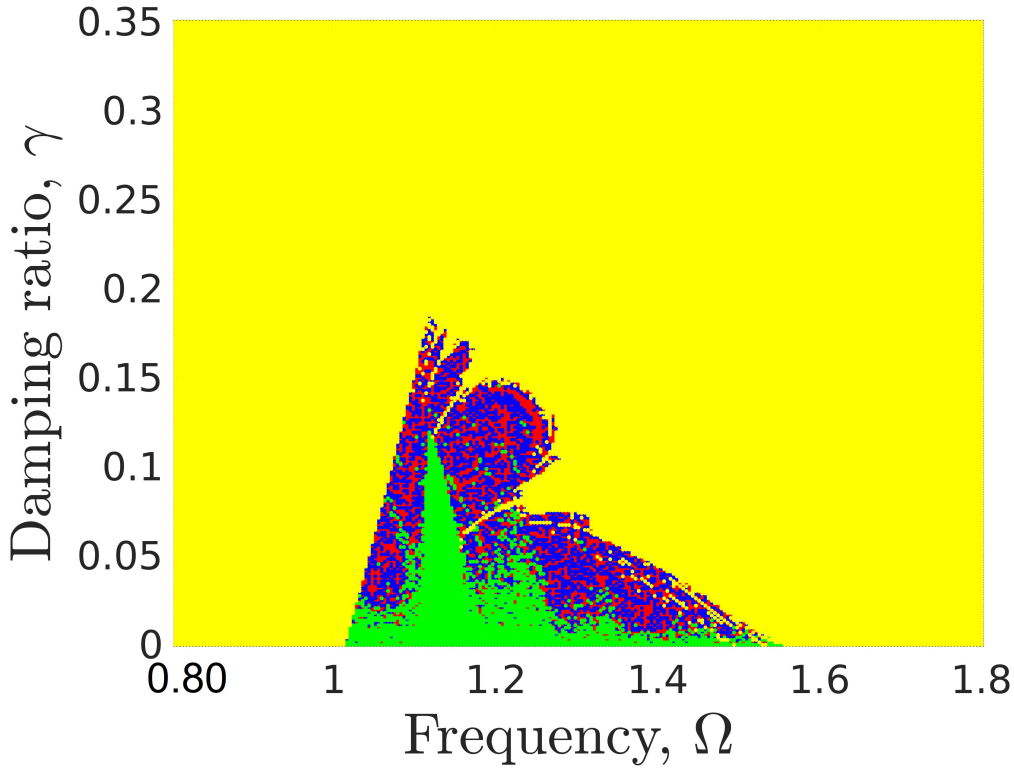


(k) $G = 0.150$

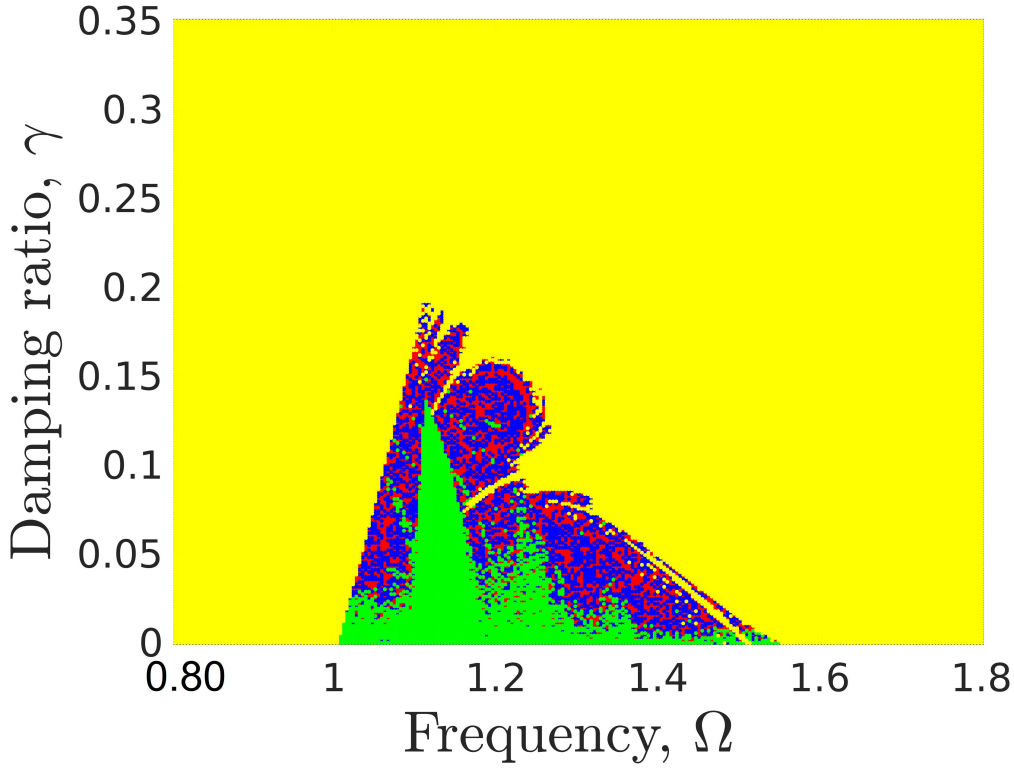


(l) $G = 0.155$

Figure S7: Damping ratio-forcing frequency parameter space

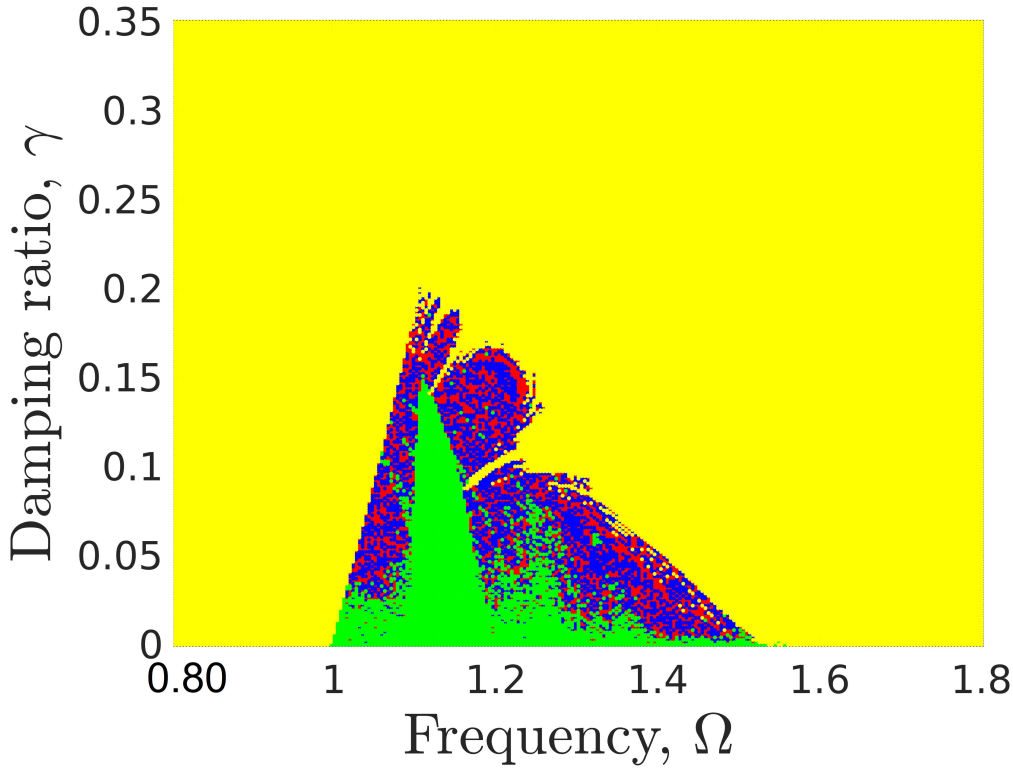


(m) $G = 0.160$

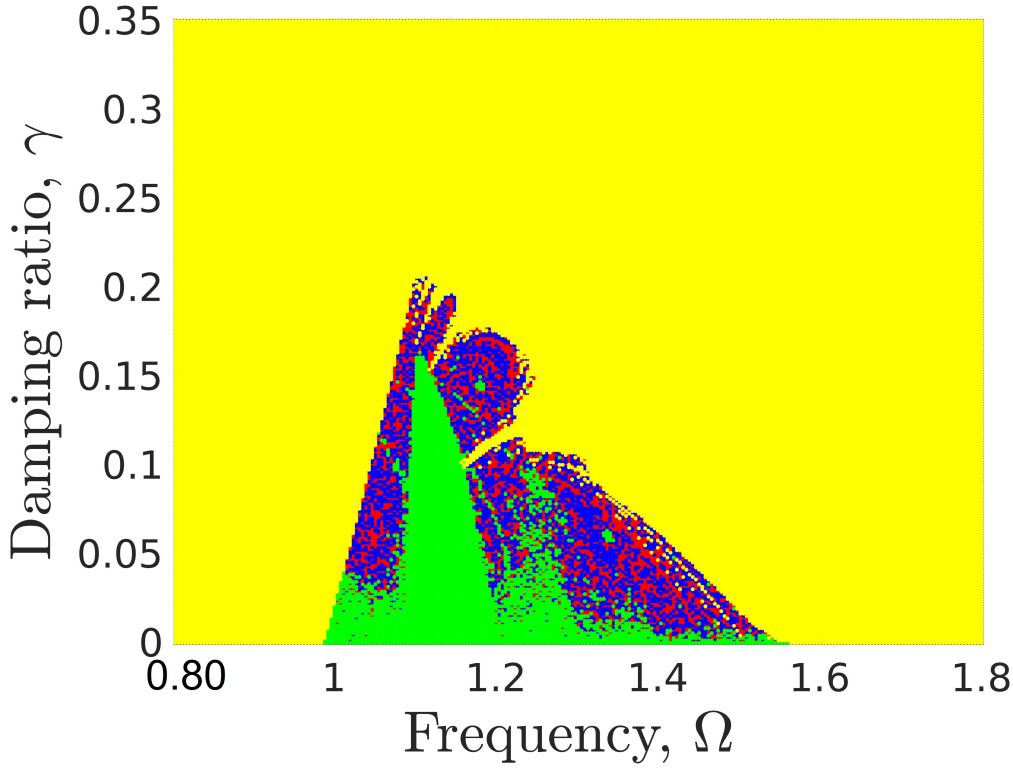


(n) $G = 0.165$

Figure S7: Damping ratio-forcing frequency parameter space

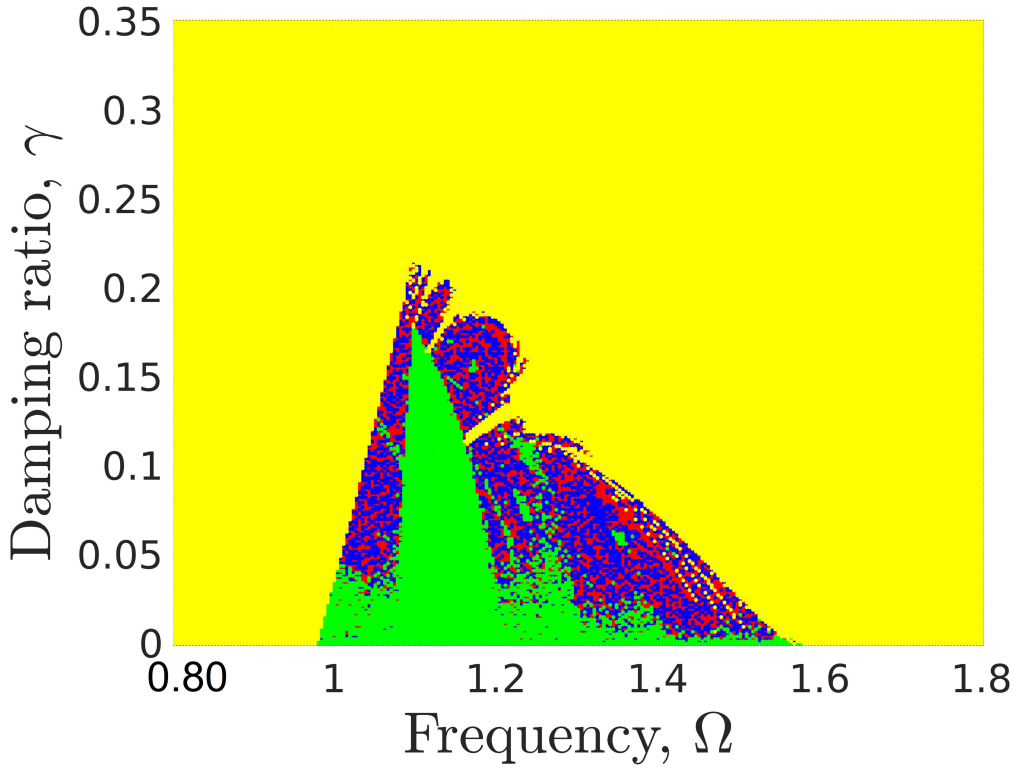


(o) $G = 0.170$

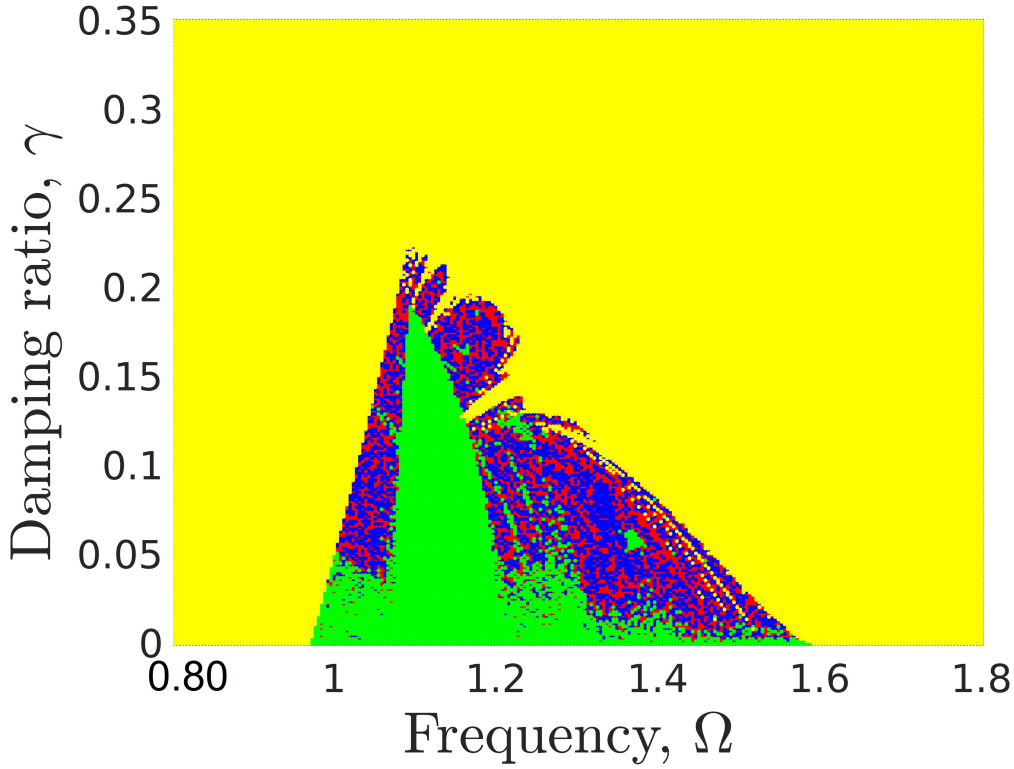


(p) $G = 0.175$

Figure S7: Damping ratio-forcing frequency parameter space

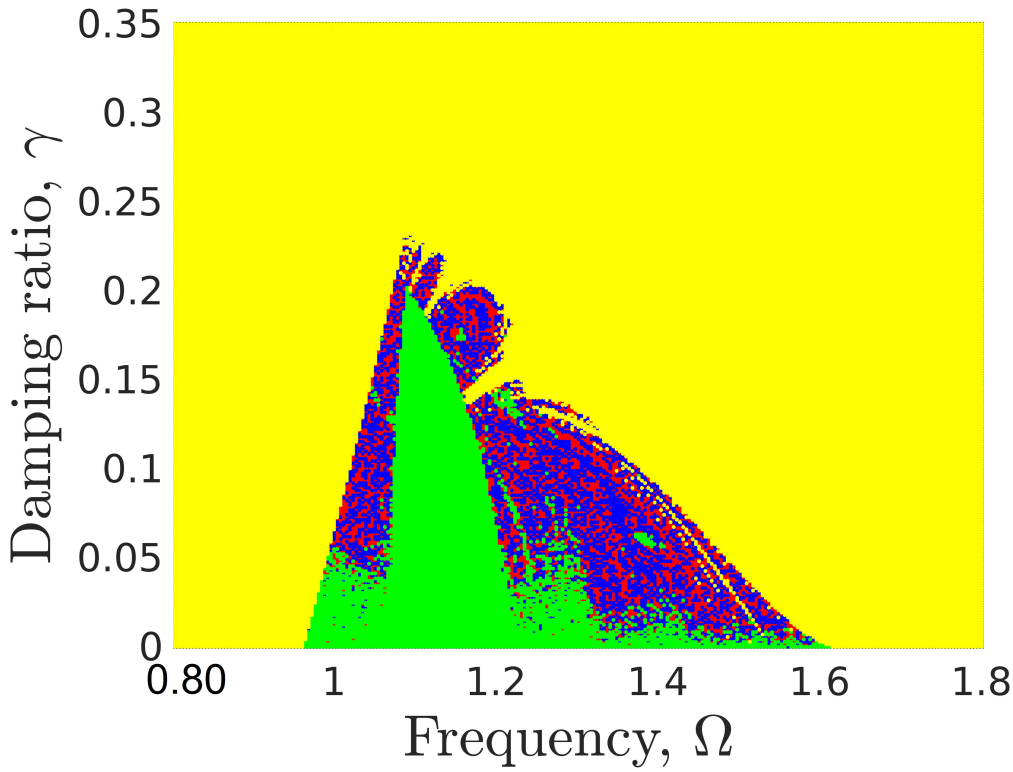


(q) $G = 0.180$

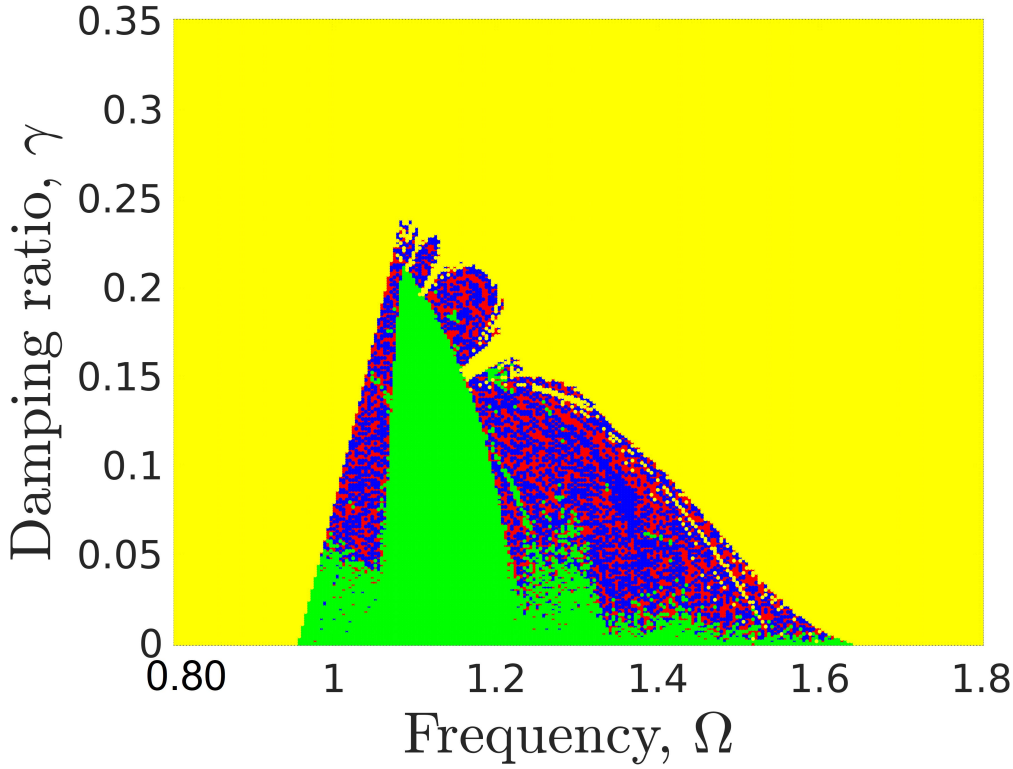


(r) $G = 0.185$

Figure S7: Damping ratio-forcing frequency parameter space

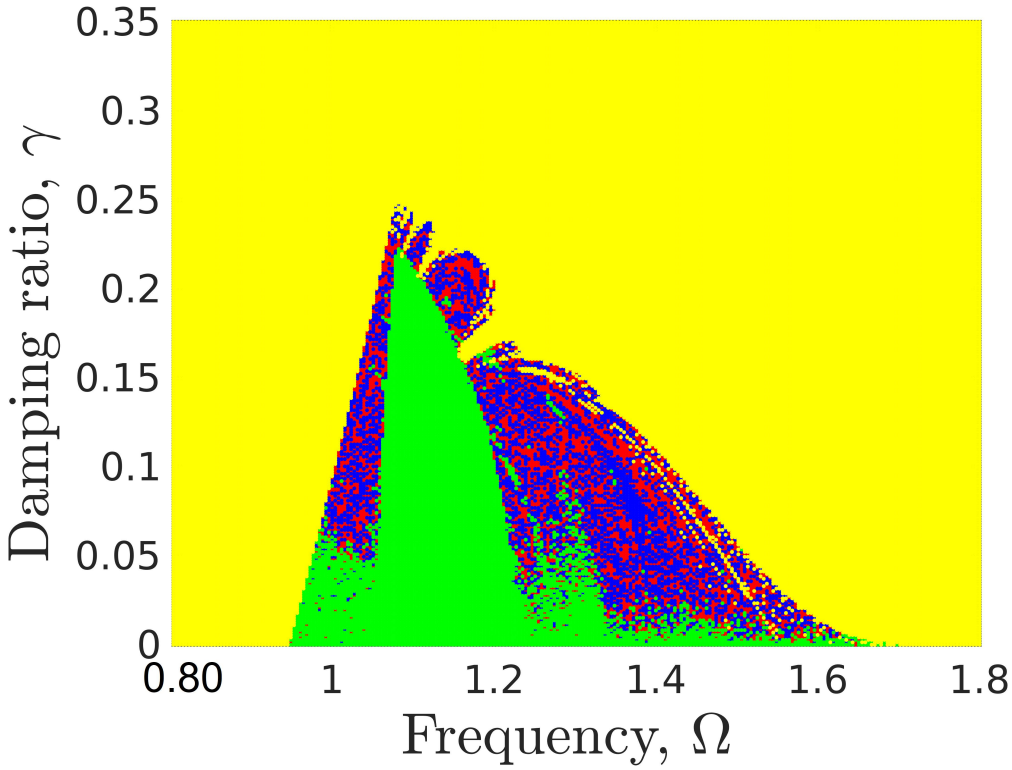


(s) $G = 0.190$

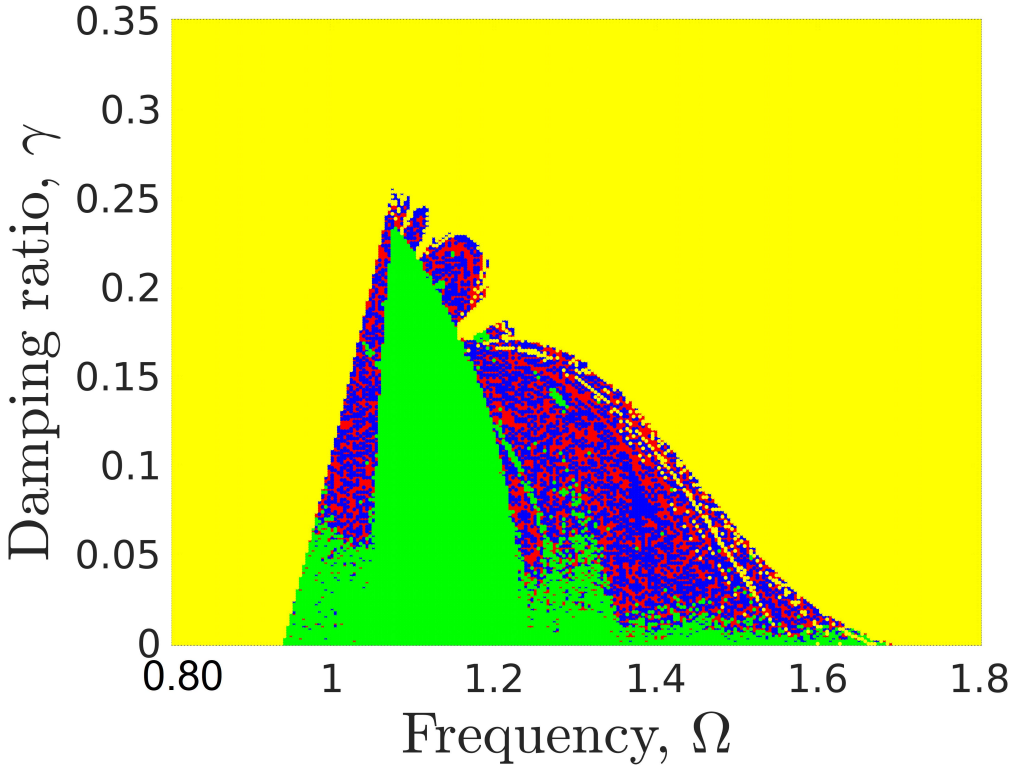


(t) $G = 0.195$

Figure S7: Damping ratio-forcing frequency parameter space

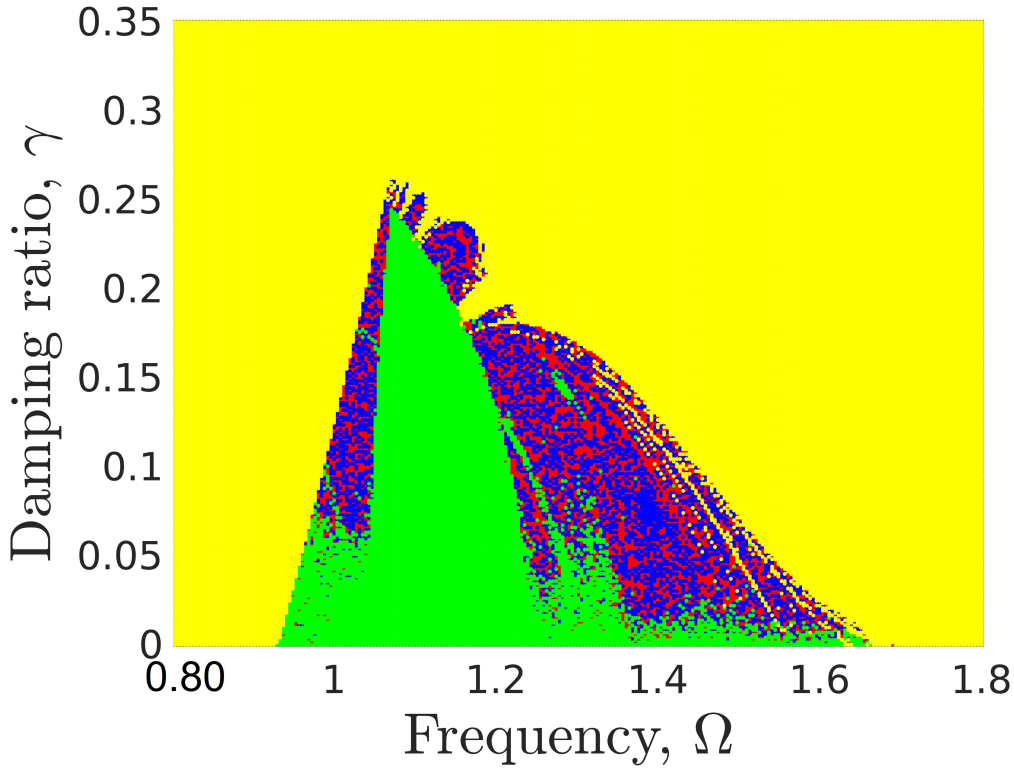


(u) $G = 0.200$

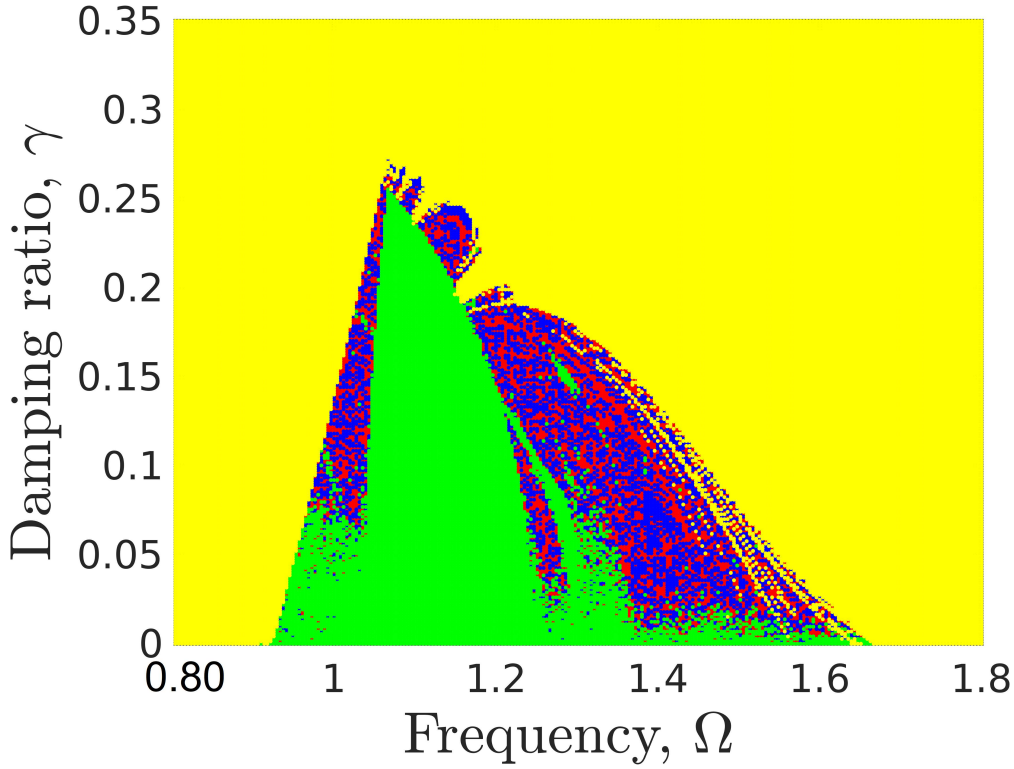


(v) $G = 0.205$

Figure S7: Damping ratio-forcing frequency parameter space

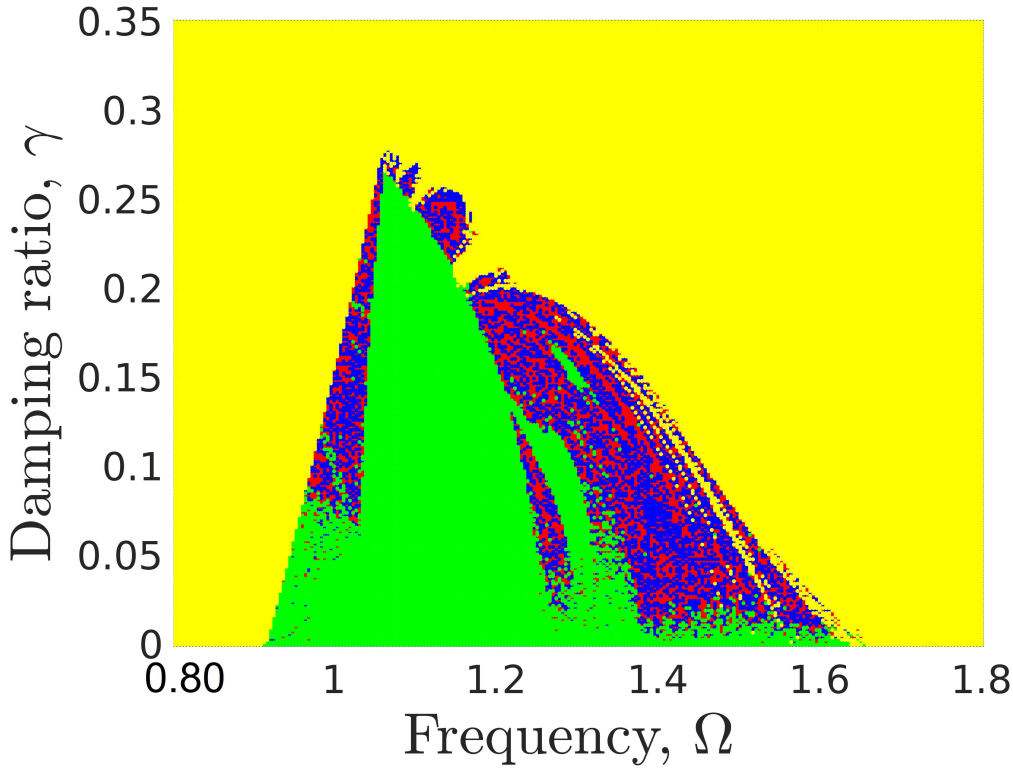


(w) $G = 0.210$

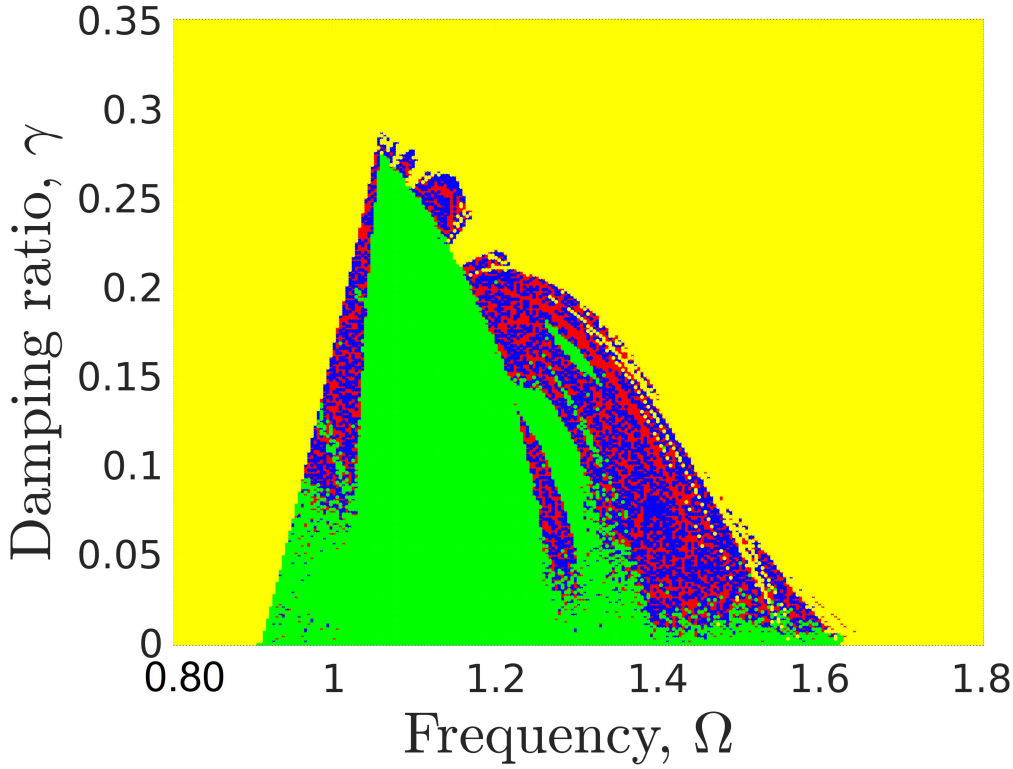


(x) $G = 0.215$

Figure S7: Damping ratio-forcing frequency parameter space

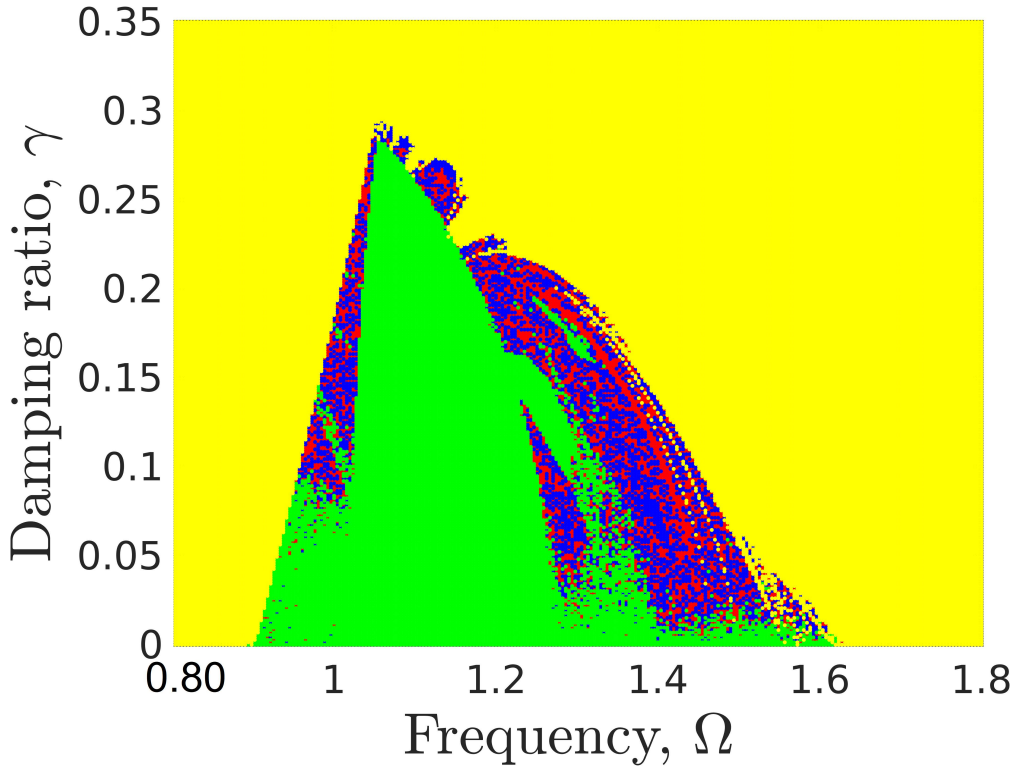


(y) $G = 0.220$

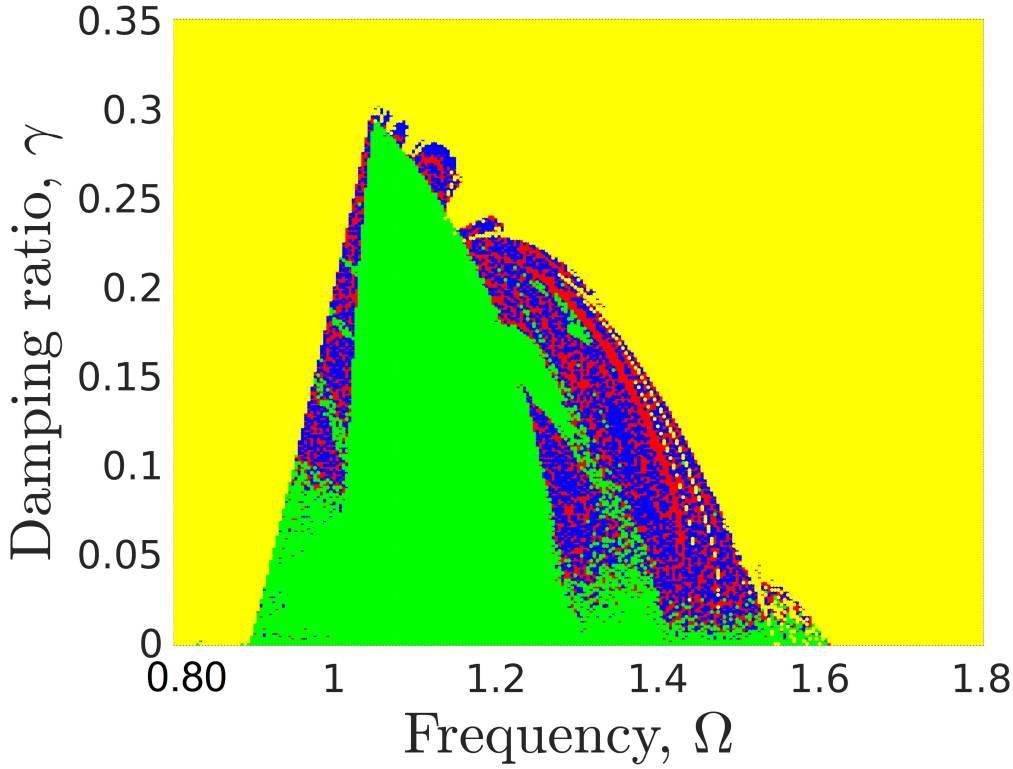


(z) $G = 0.225$

Figure S7: Damping ratio-forcing frequency parameter space

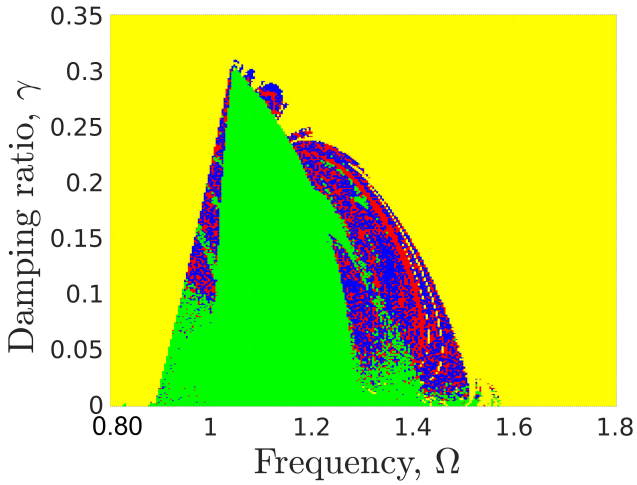


(aa) $G = 0.230$

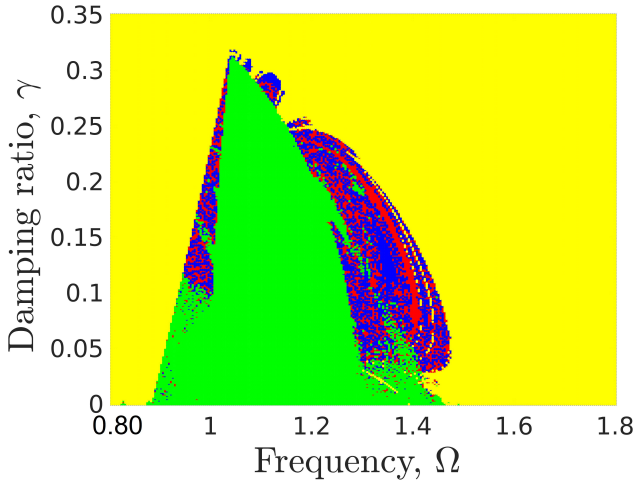


(ab) $G = 0.235$

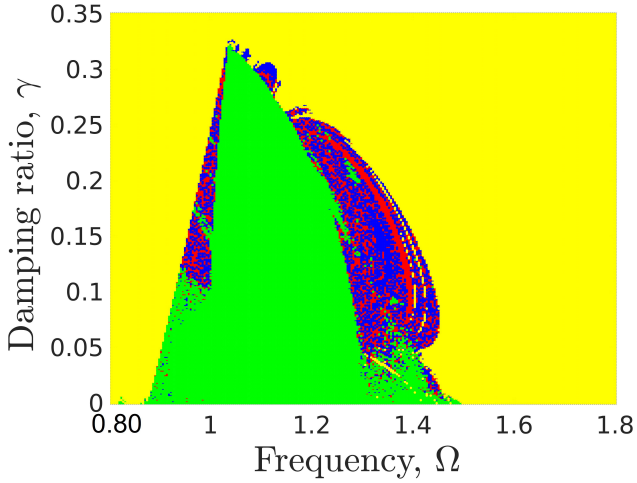
Figure S7: Damping ratio-forcing frequency parameter space



(ac) $G = 0.240$



(ad) $G = 0.245$

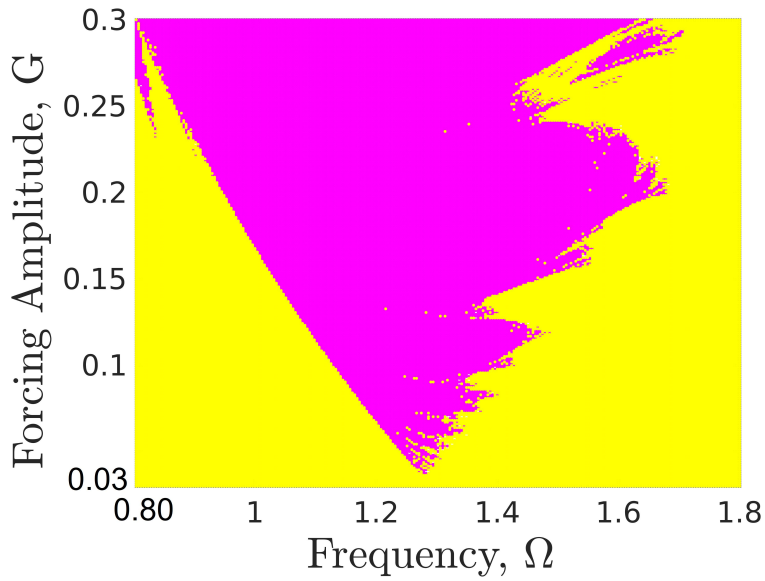


(ae) $G = 0.250$

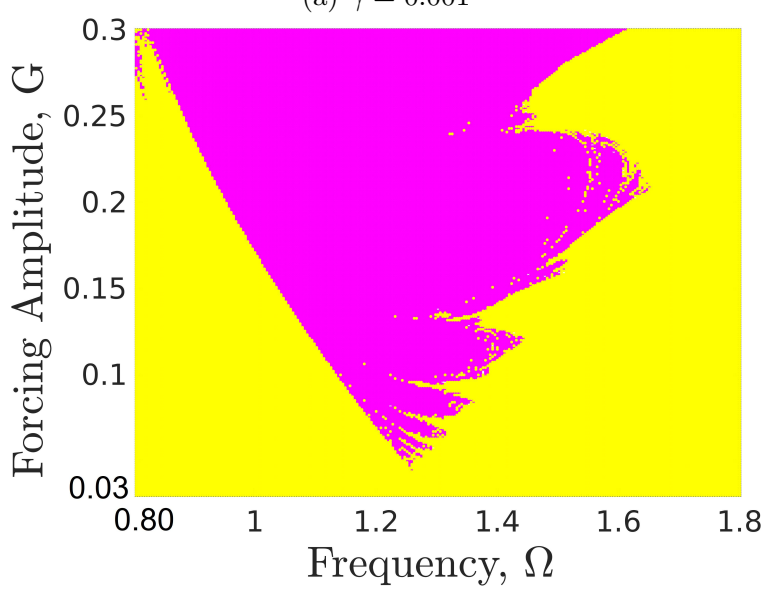
Figure S7: Damping ratio-forcing frequency parameter space

S6 Two-Colored Forcing Amplitude-Frequency (G vs. Ω) Parameter Space

31 two-colored forcing amplitude-frequency parameter space for γ ranges from $0.001 \sim 0.30$ with the increment of 0.01.

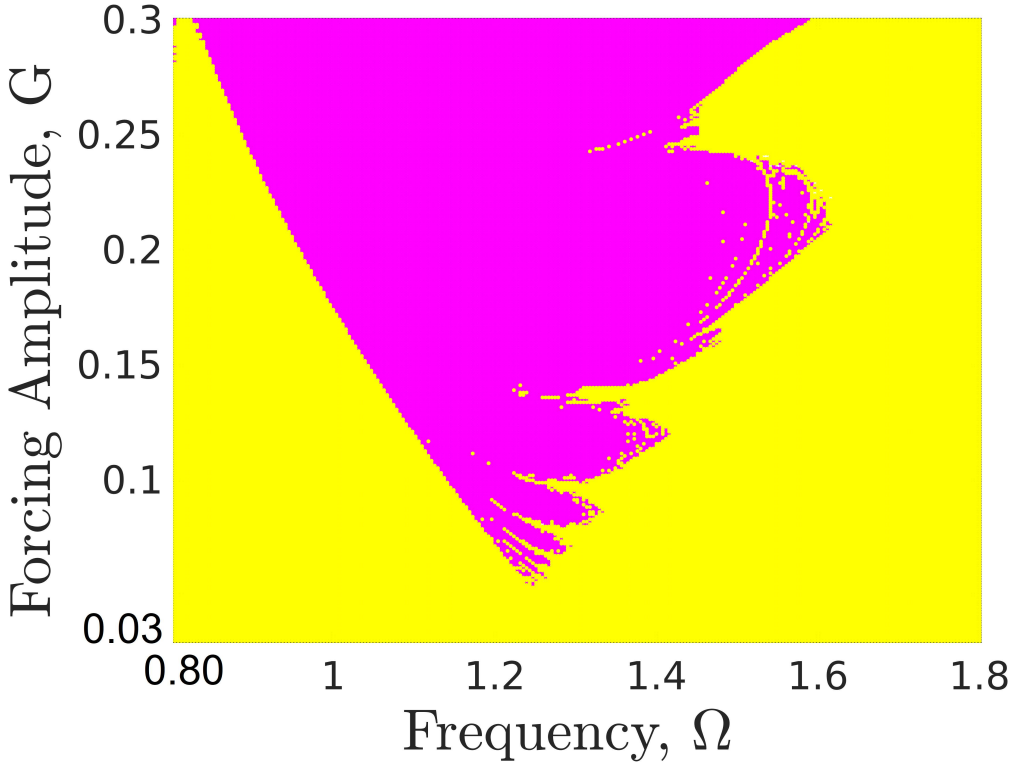


(a) $\gamma = 0.001$

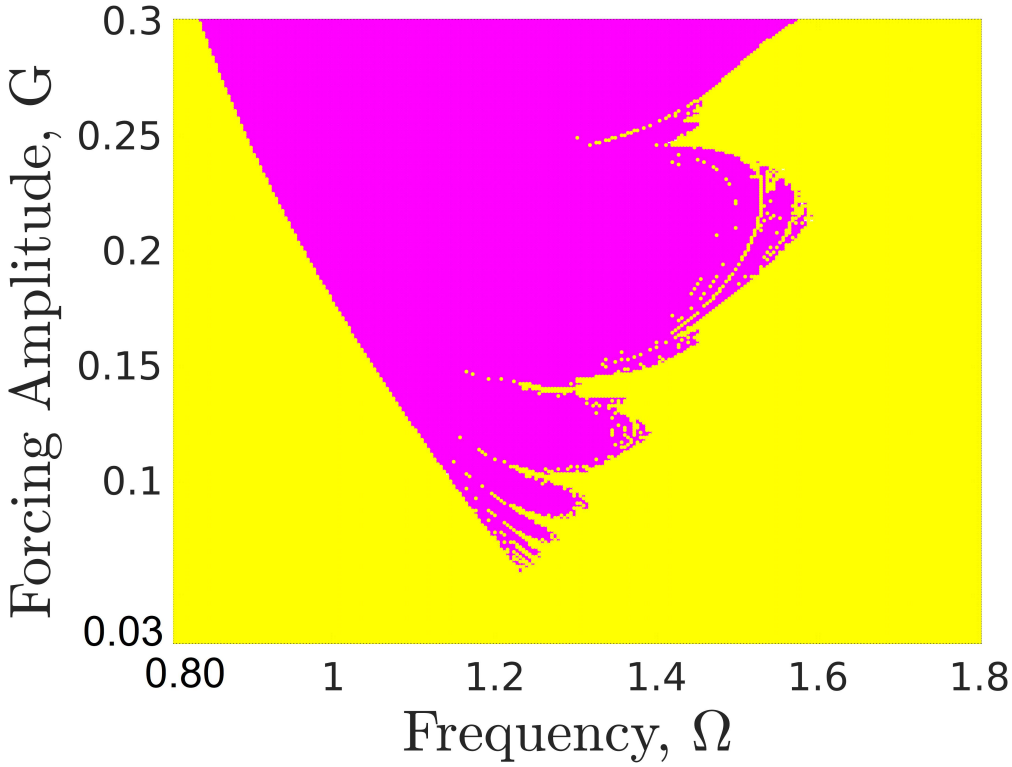


(b) $\gamma = 0.01$

Figure S8: Two-colored forcing amplitude-frequency parameter space

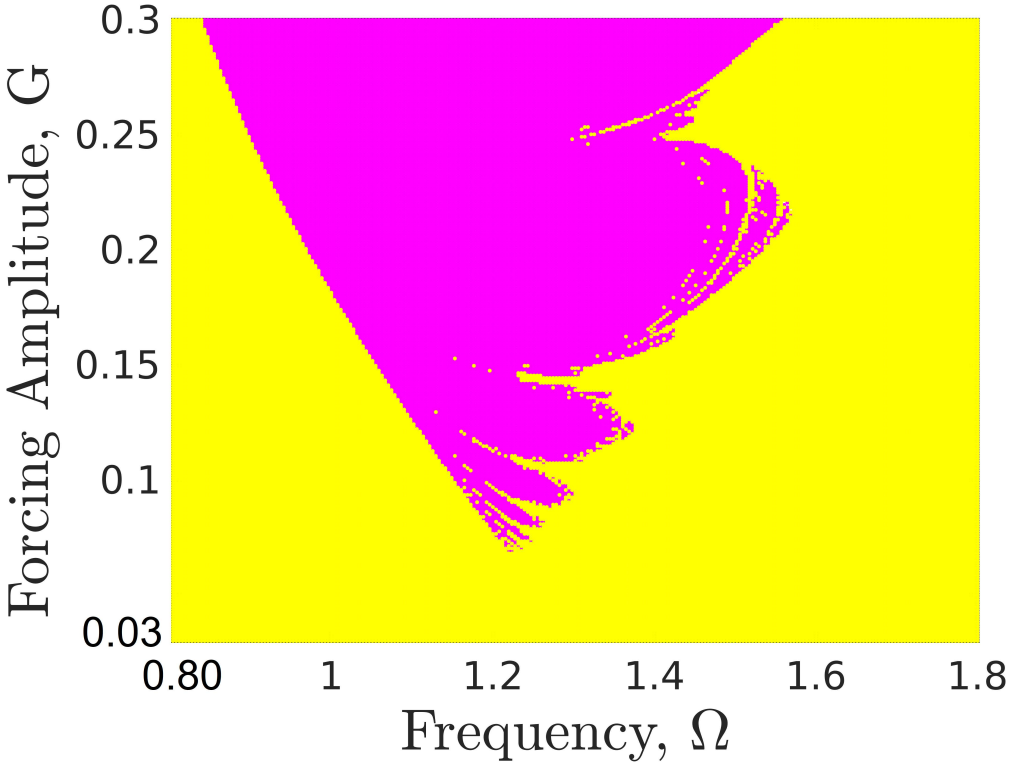


(c) $\gamma = 0.02$

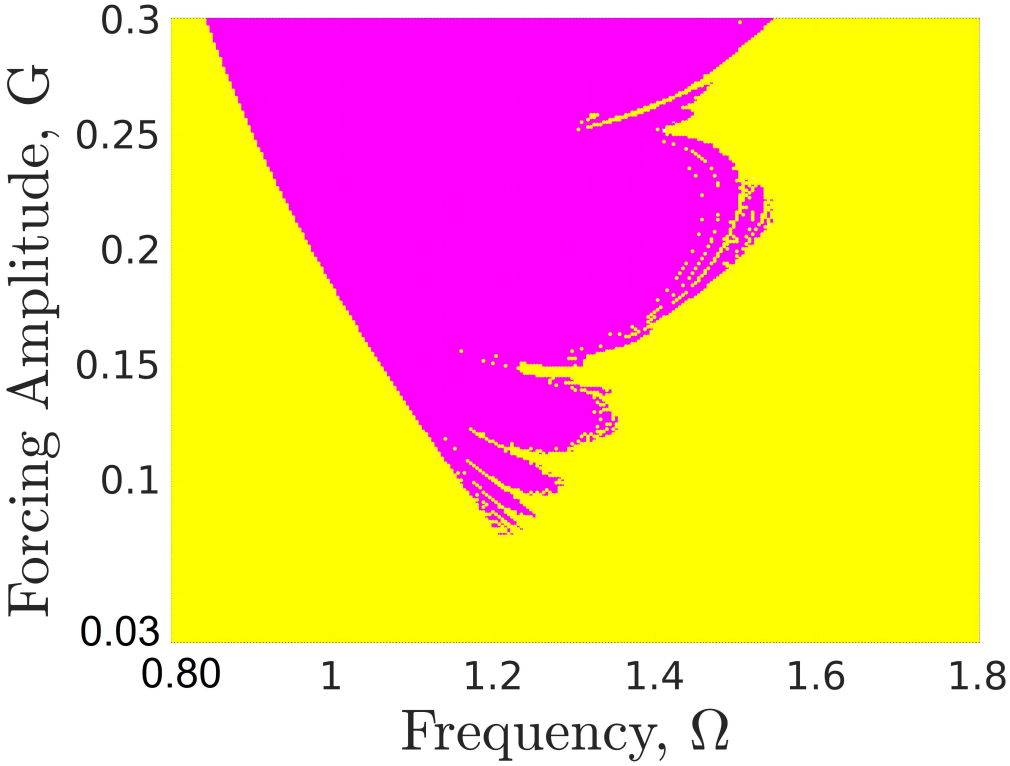


(d) $\gamma = 0.03$

Figure S8: Two-colored forcing amplitude-frequency parameter space

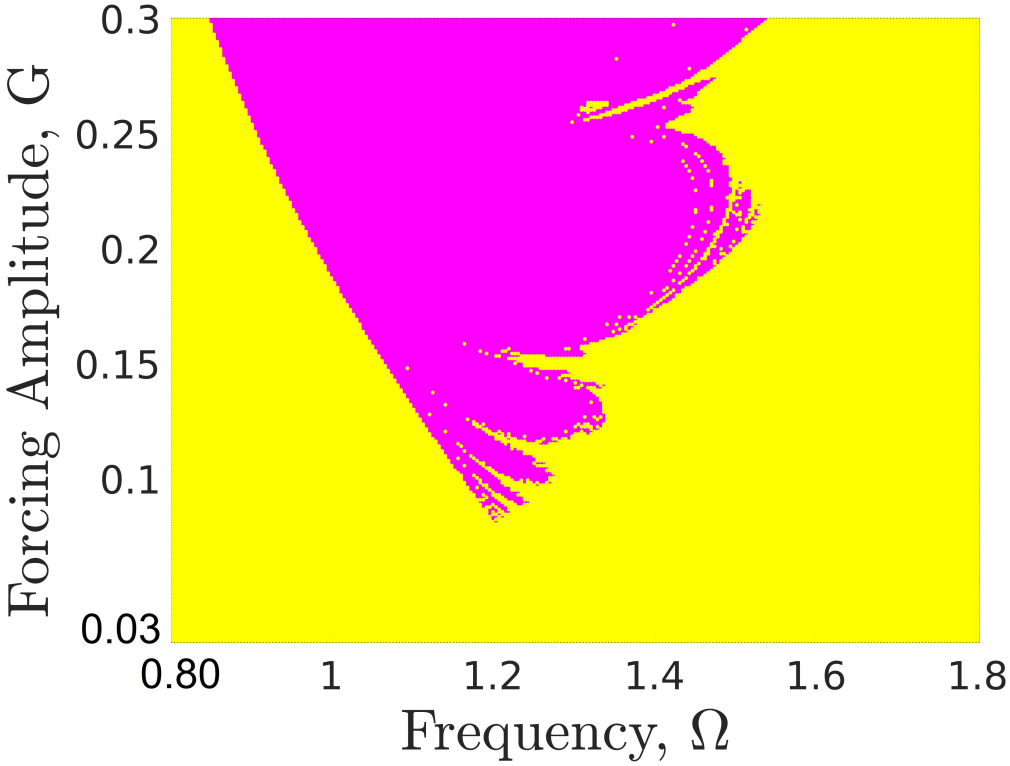


(e) $\gamma = 0.04$

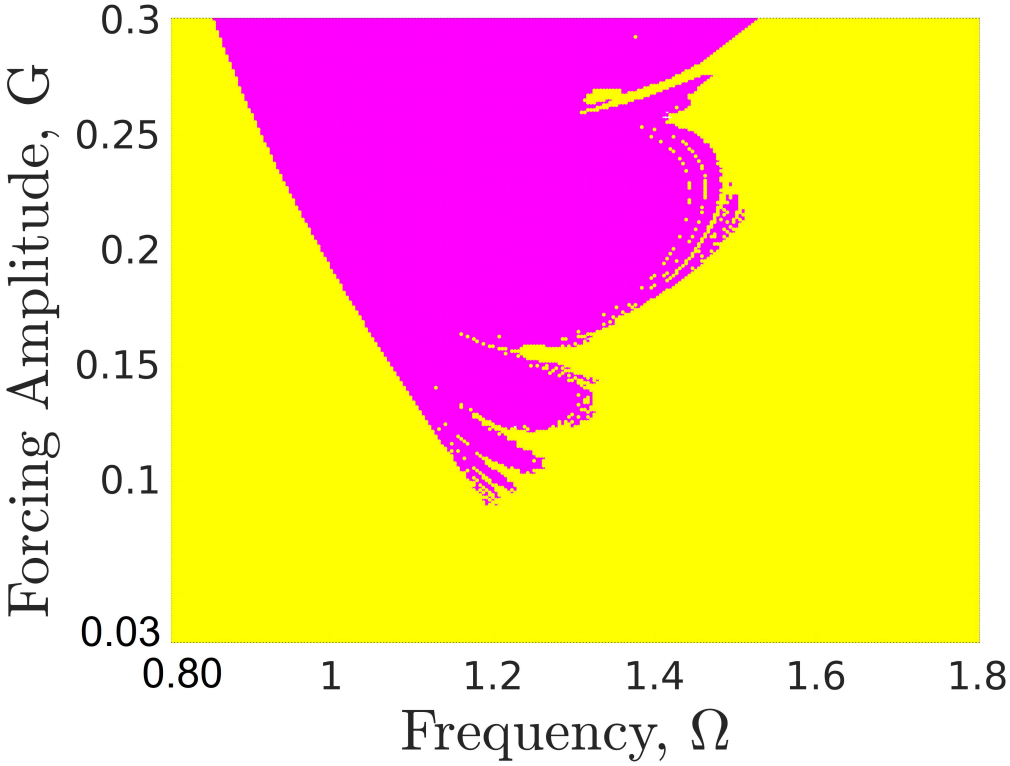


(f) $\gamma = 0.05$

Figure S8: Two-colored forcing amplitude-frequency parameter space

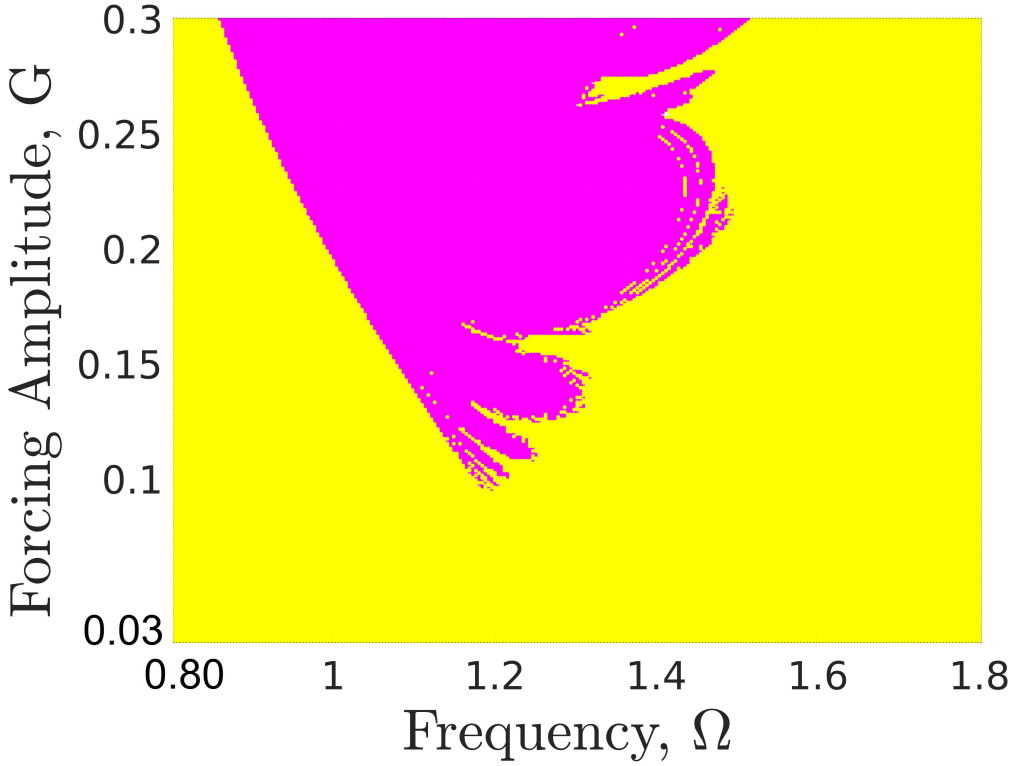


(g) $\gamma = 0.06$

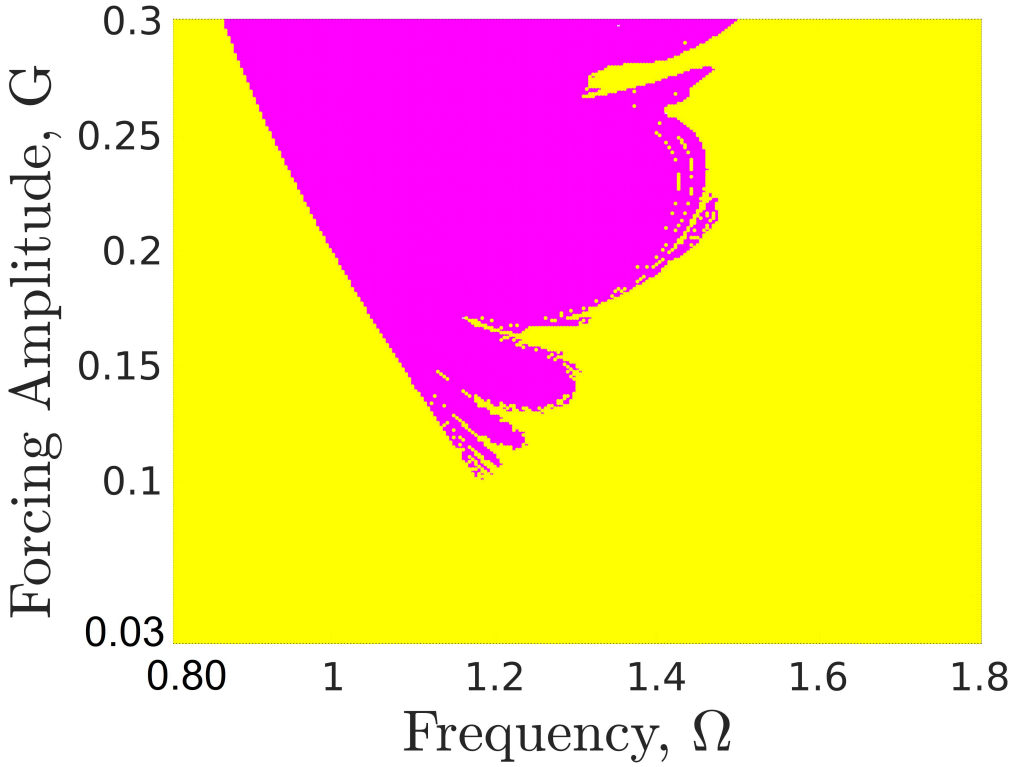


(h) $\gamma = 0.07$

Figure S8: Two-colored forcing amplitude-frequency parameter space

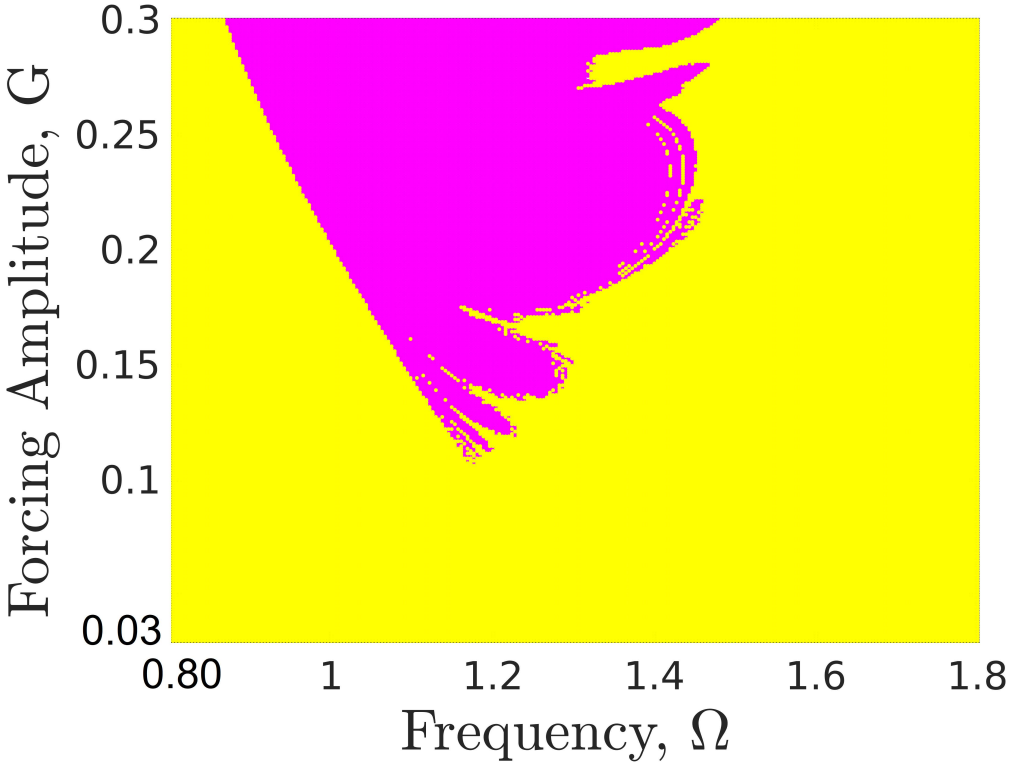


(i) $\gamma = 0.08$

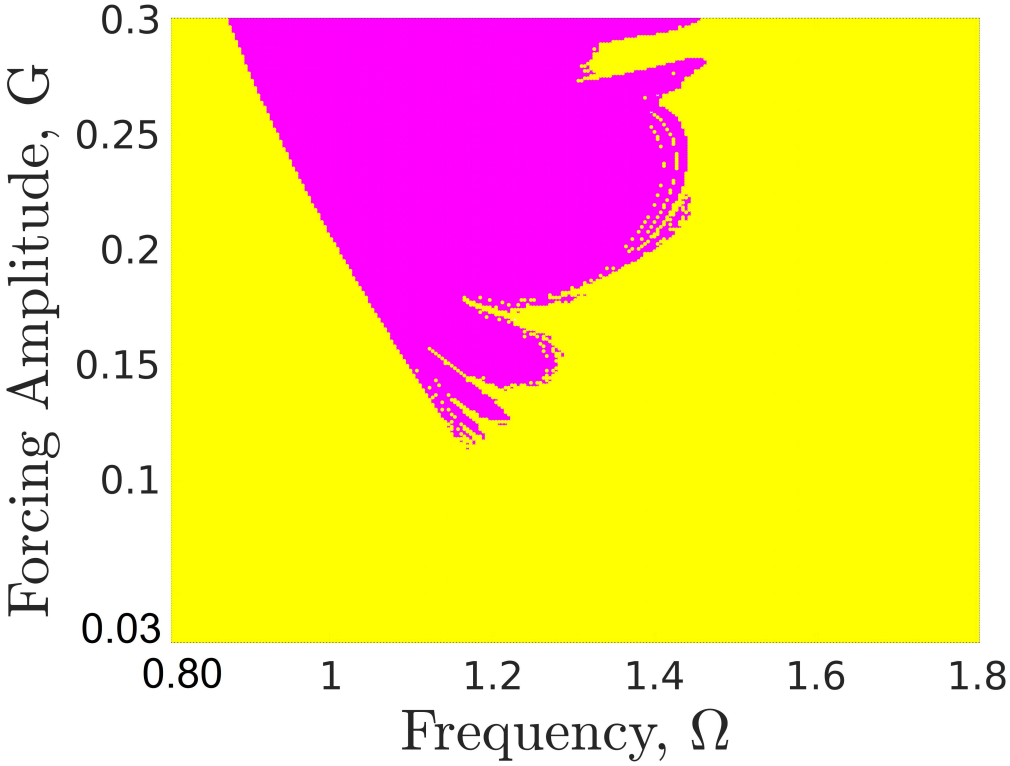


(j) $\gamma = 0.09$

Figure S8: Two-colored forcing amplitude-frequency parameter space

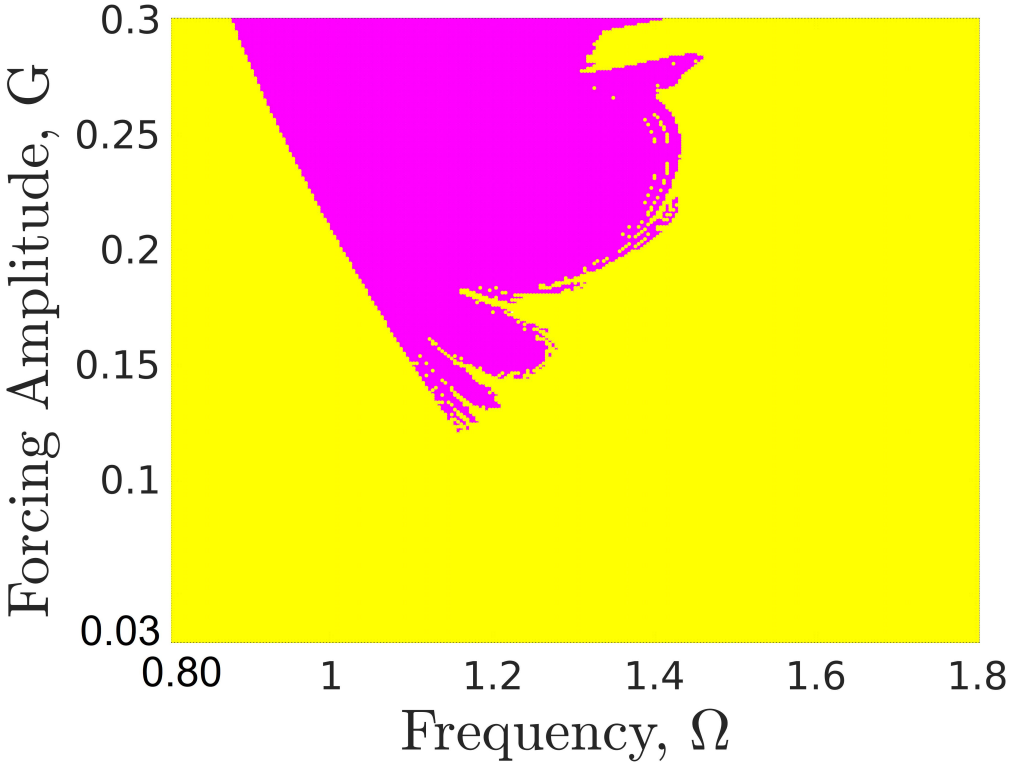


(k) $\gamma = 0.10$

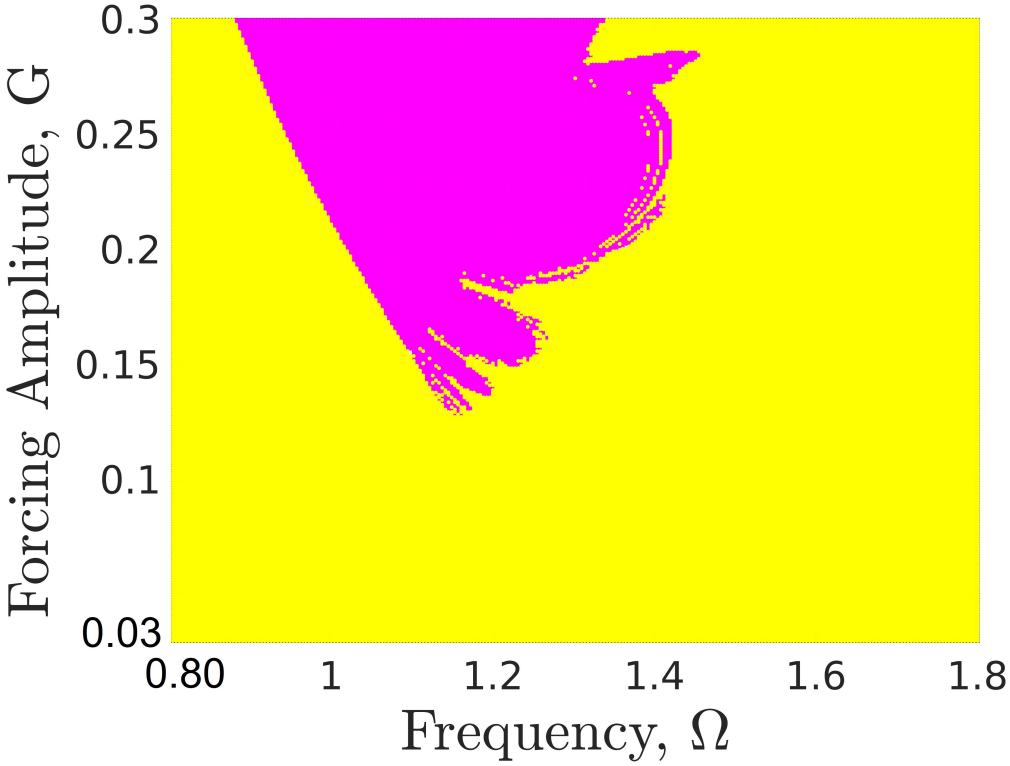


(l) $\gamma = 0.11$

Figure S8: Two-colored forcing amplitude-frequency parameter space

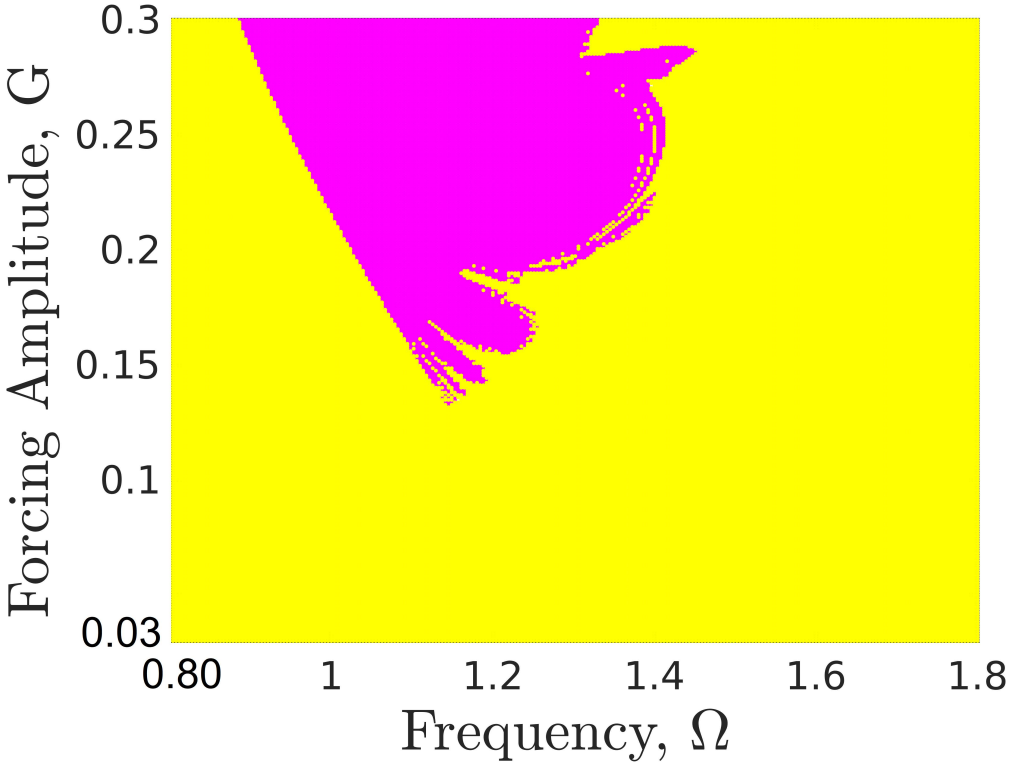


(m) $\gamma = 0.12$

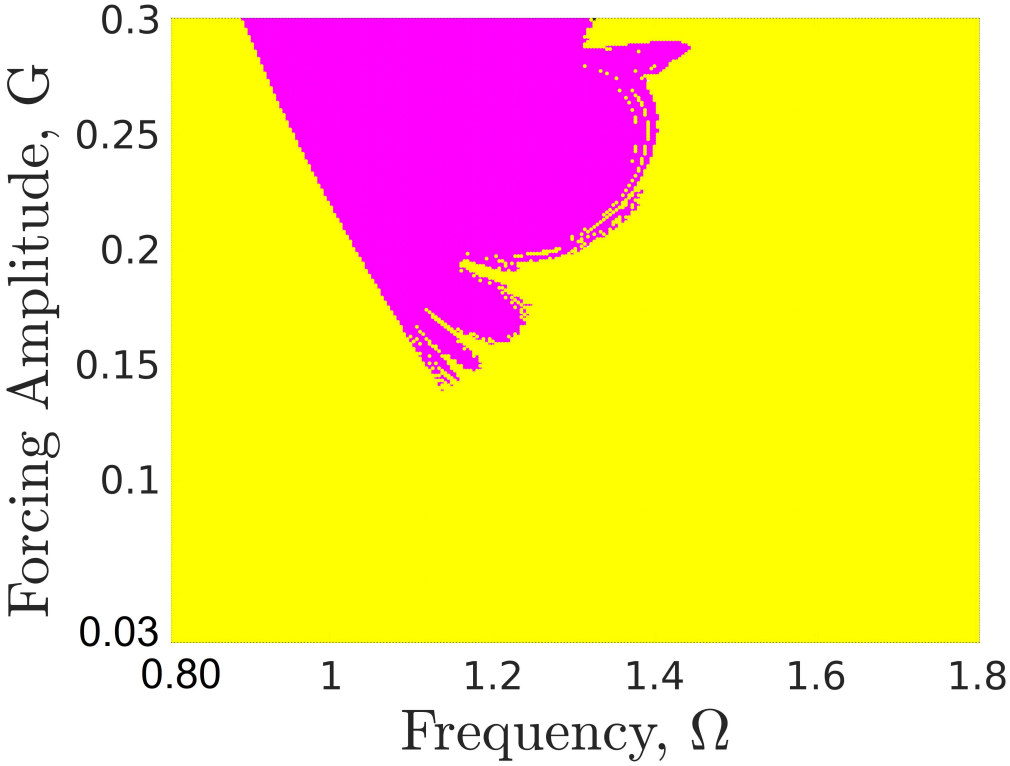


(n) $\gamma = 0.13$

Figure S8: Two-colored forcing amplitude-frequency parameter space

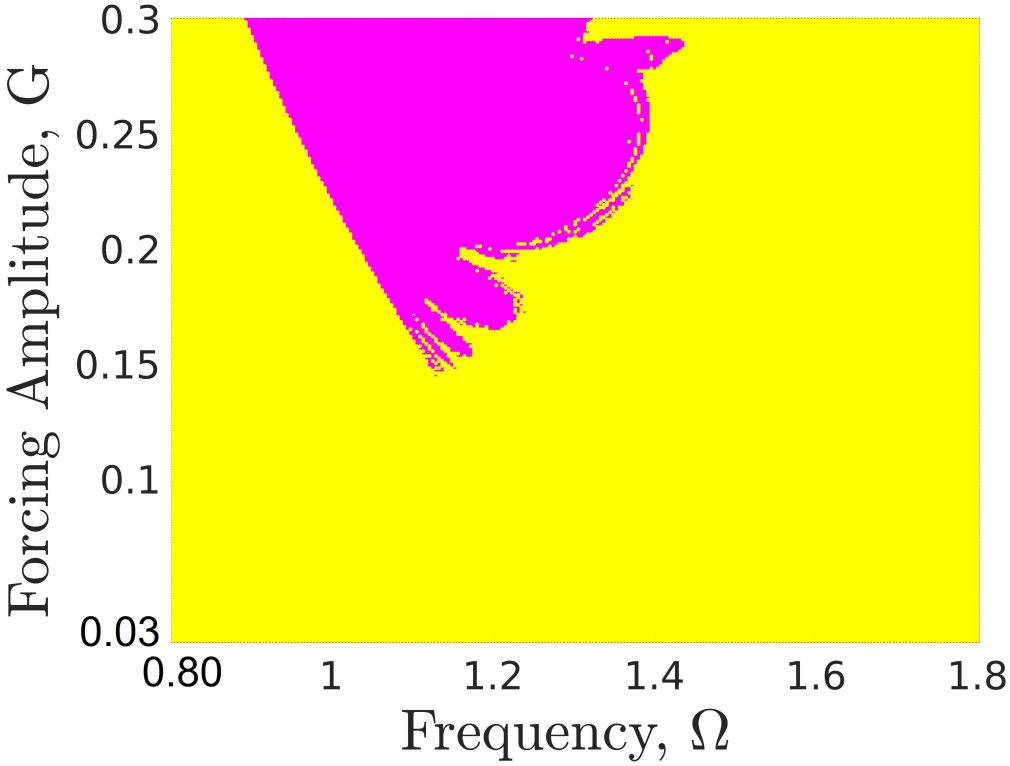


(o) $\gamma = 0.14$

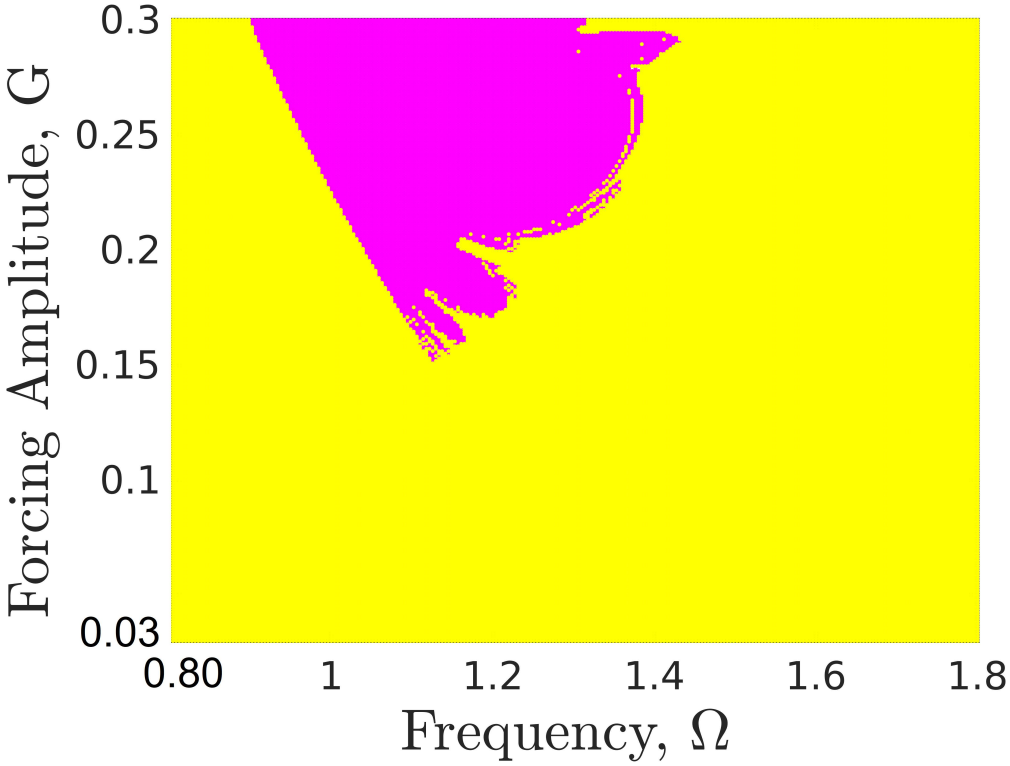


(p) $\gamma = 0.15$

Figure S8: Two-colored forcing amplitude-frequency parameter space

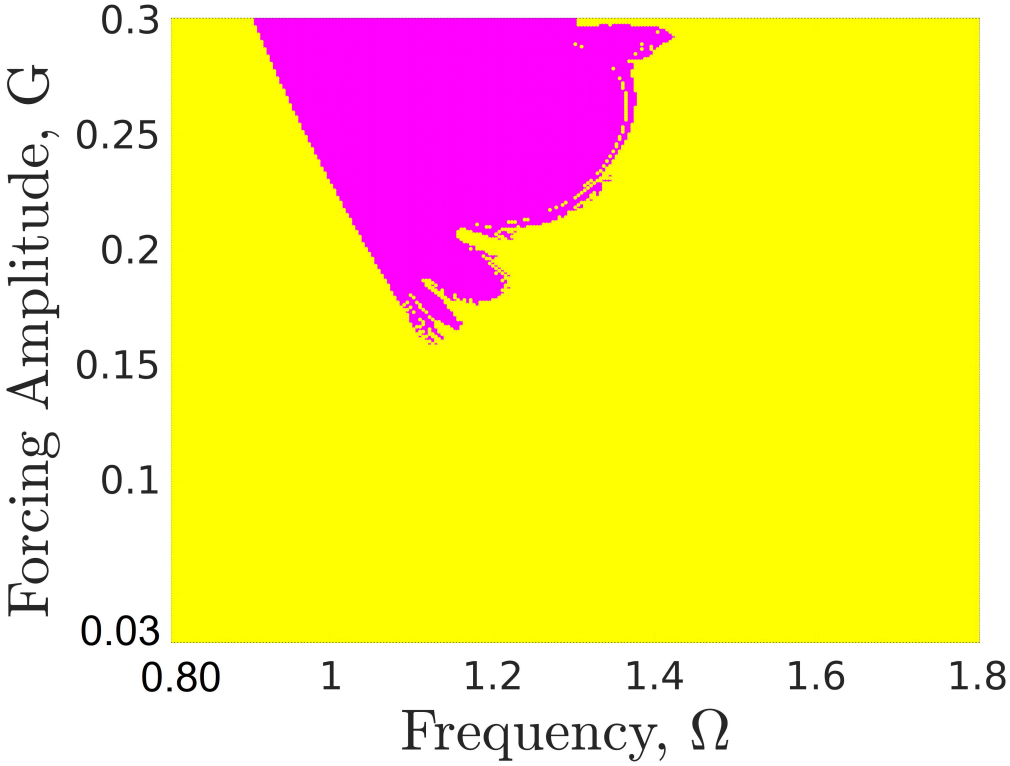


(q) $\gamma = 0.16$

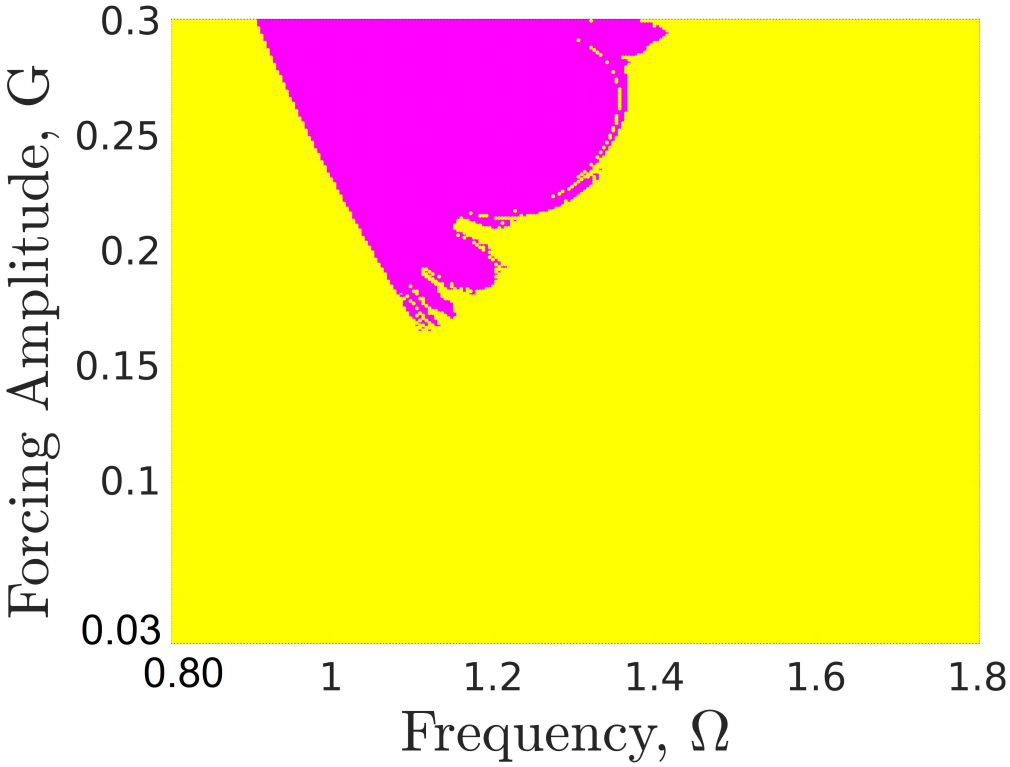


(r) $\gamma = 0.17$

Figure S8: Two-colored forcing amplitude-frequency parameter space

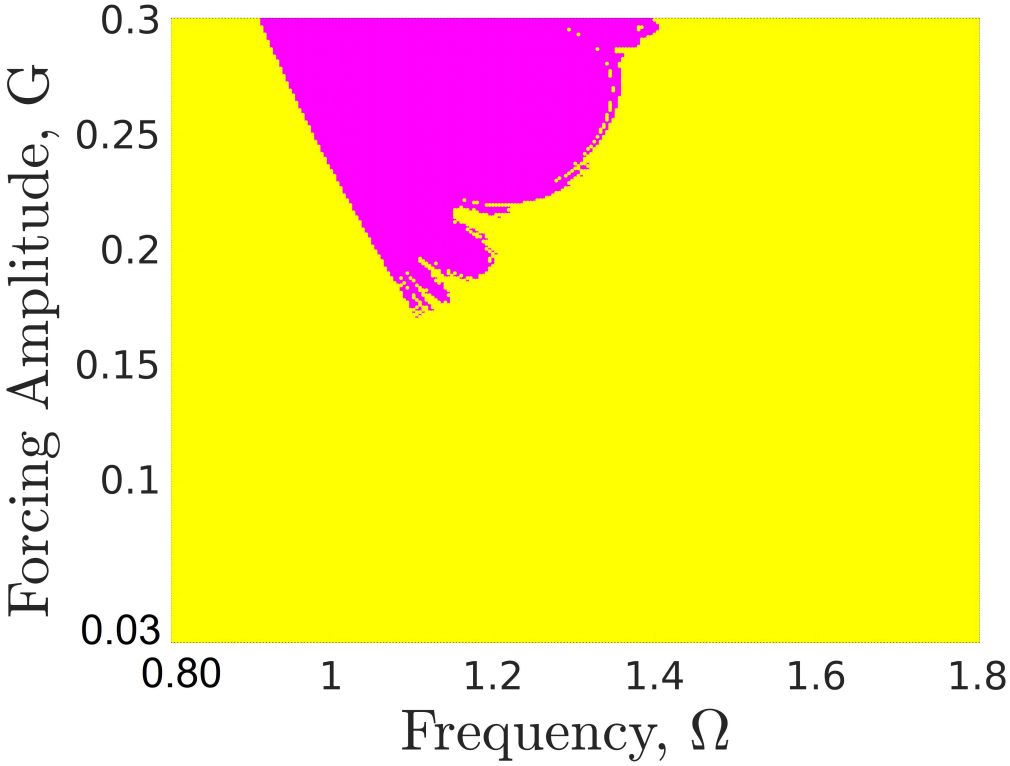


(s) $\gamma = 0.18$

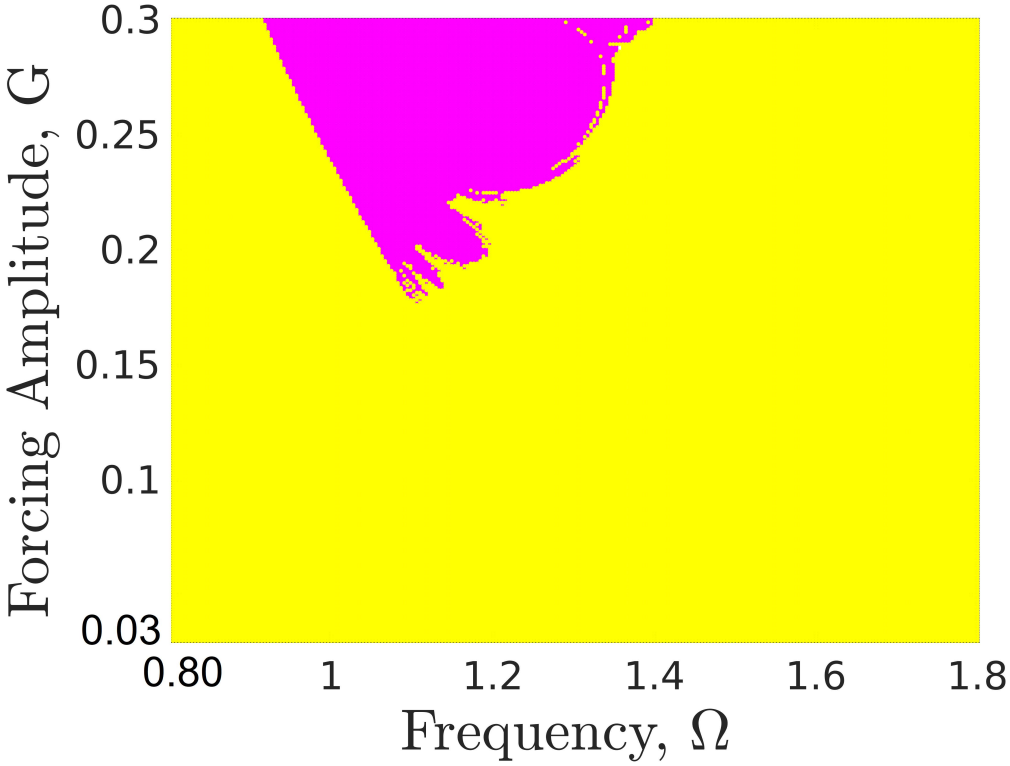


(t) $\gamma = 0.19$

Figure S8: Two-colored forcing amplitude-frequency parameter space

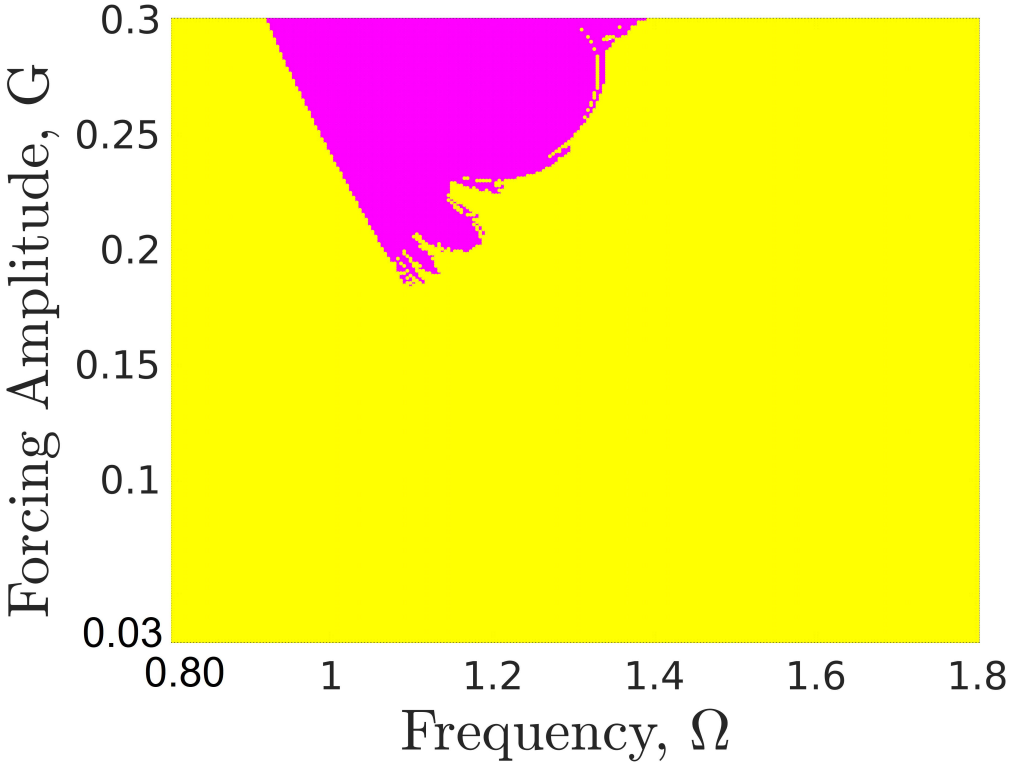


(u) $\gamma = 0.20$

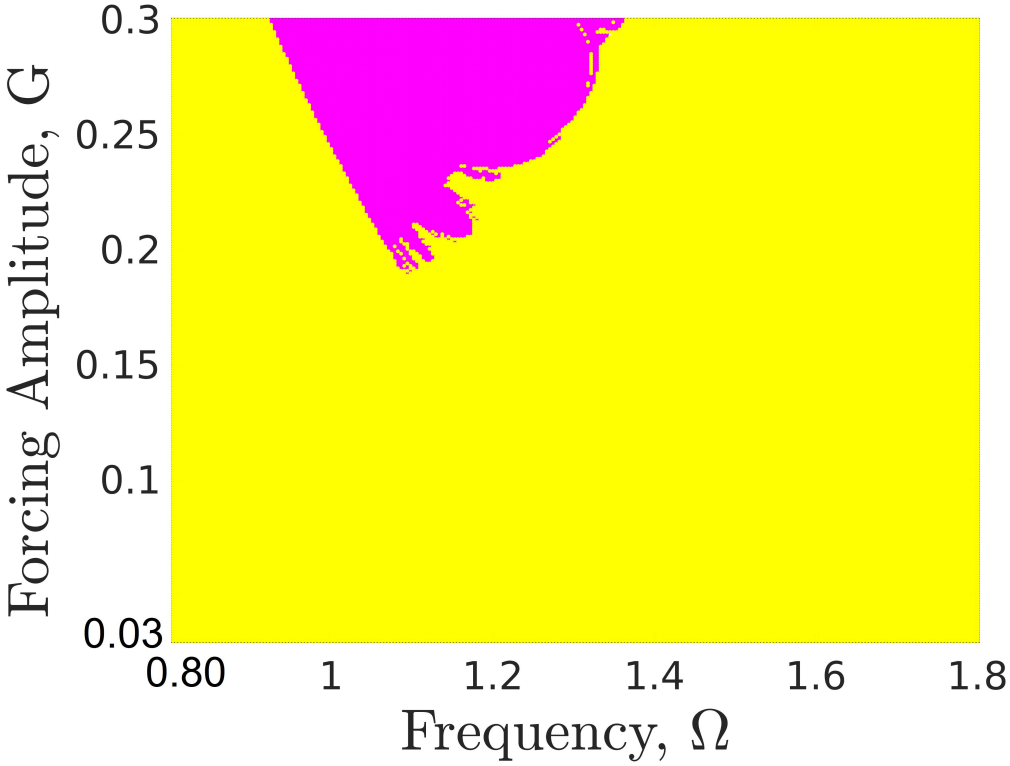


(v) $\gamma = 0.21$

Figure S8: Two-colored forcing amplitude-frequency parameter space

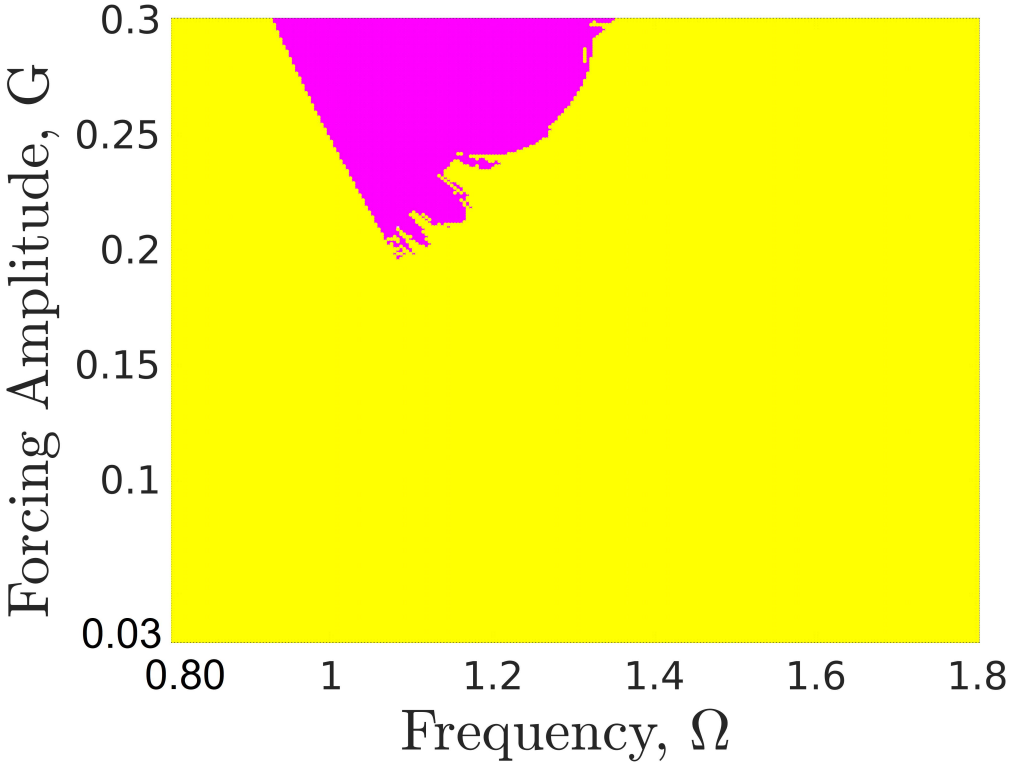


(w) $\gamma = 0.22$

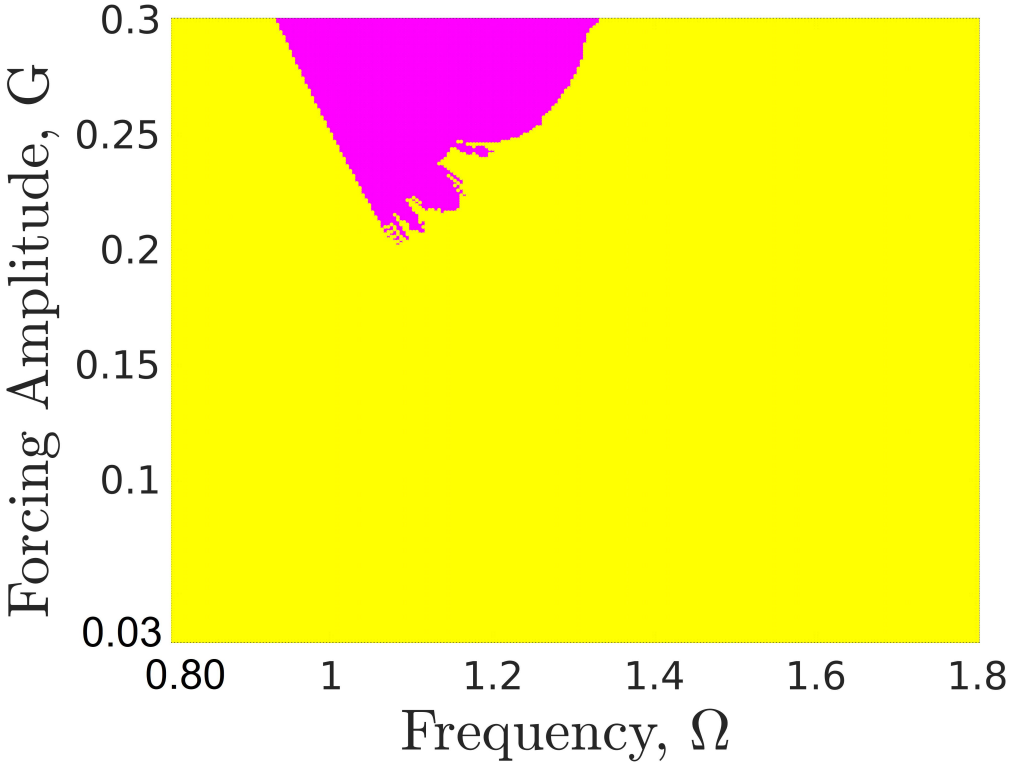


(x) $\gamma = 0.23$

Figure S8: Two-colored forcing amplitude-frequency parameter space

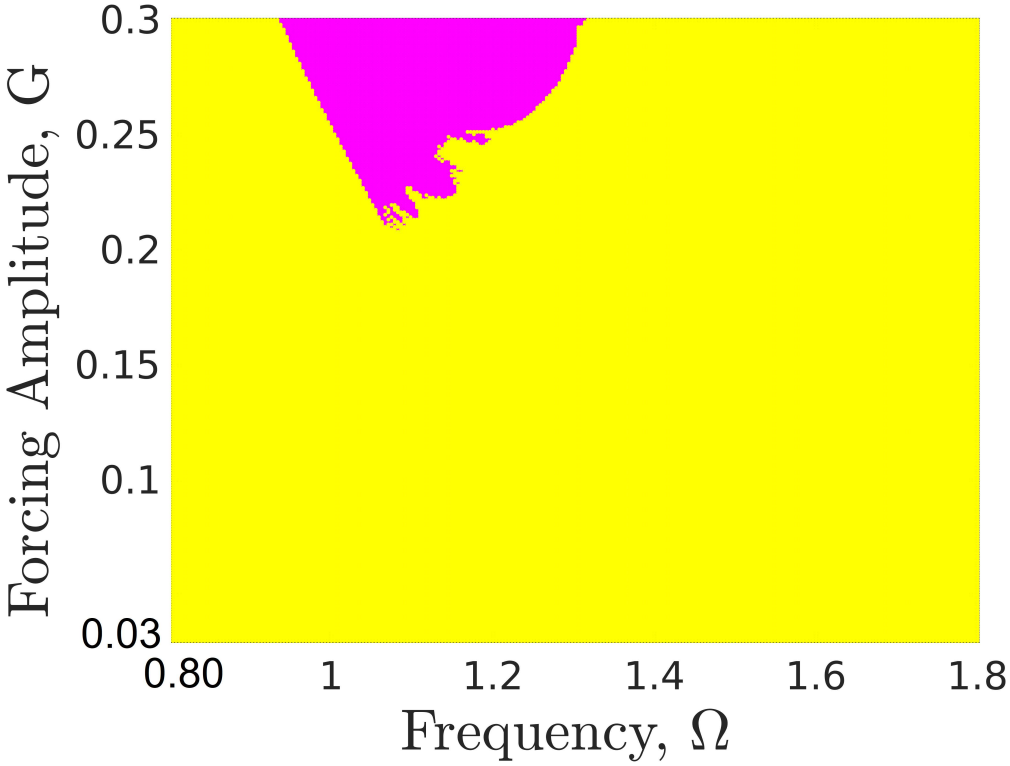


(y) $\gamma = 0.24$

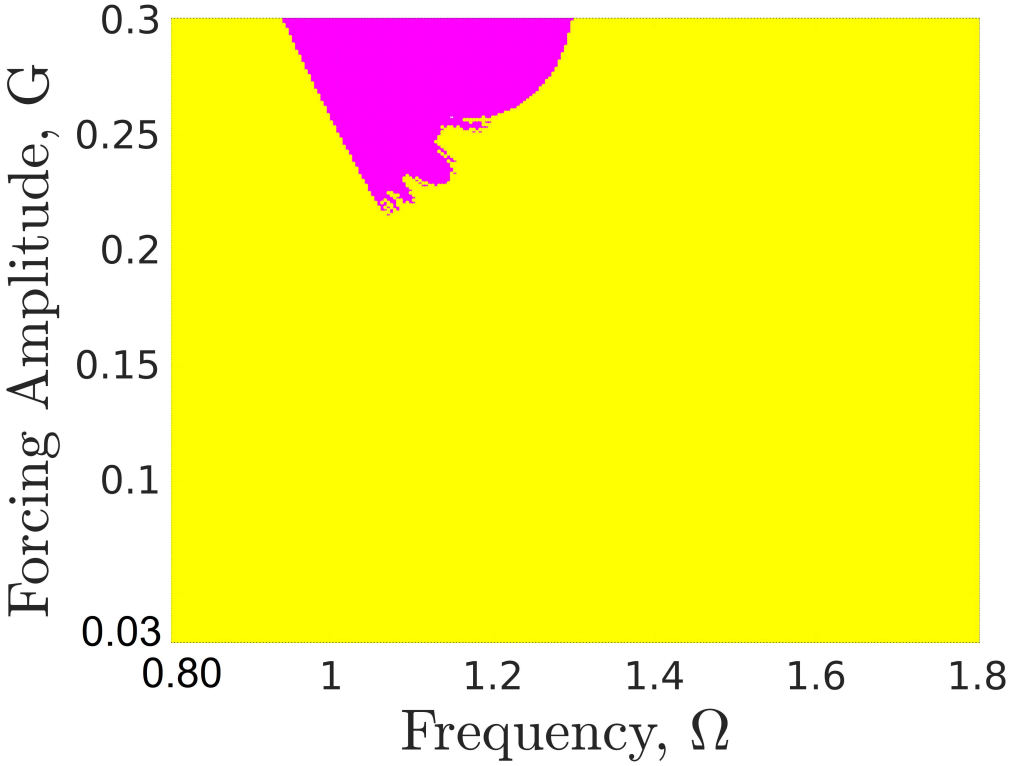


(z) $\gamma = 0.25$

Figure S8: Two-colored forcing amplitude-frequency parameter space

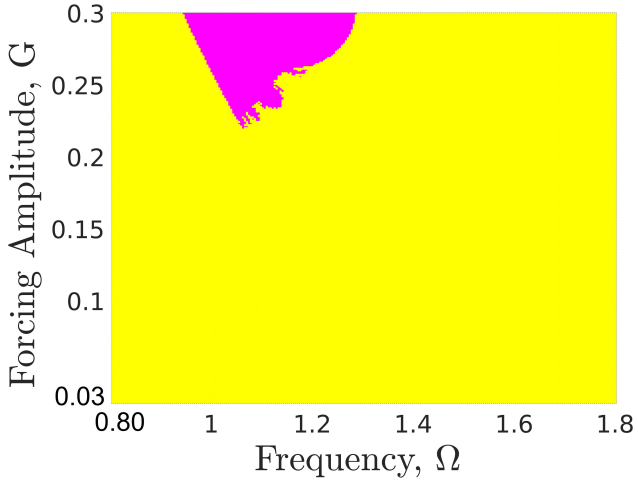


(aa) $\gamma = 0.26$

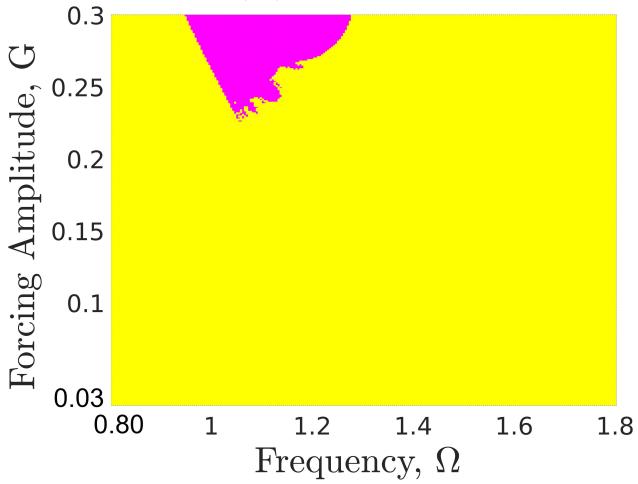


(ab) $\gamma = 0.27$

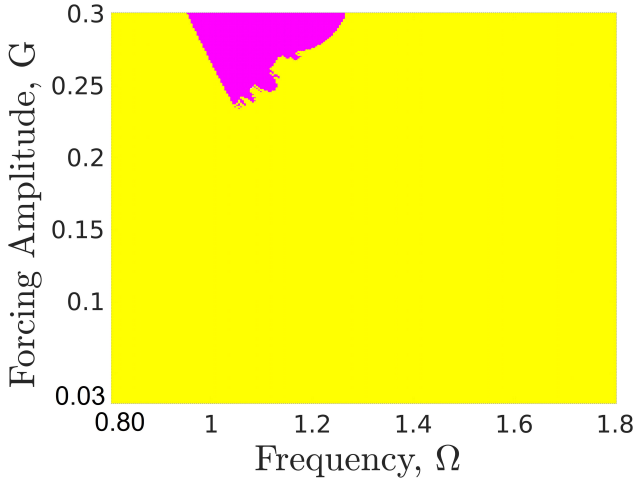
Figure S8: Two-colored forcing amplitude-frequency parameter space



(ac) $\gamma = 0.28$



(ad) $\gamma = 0.29$

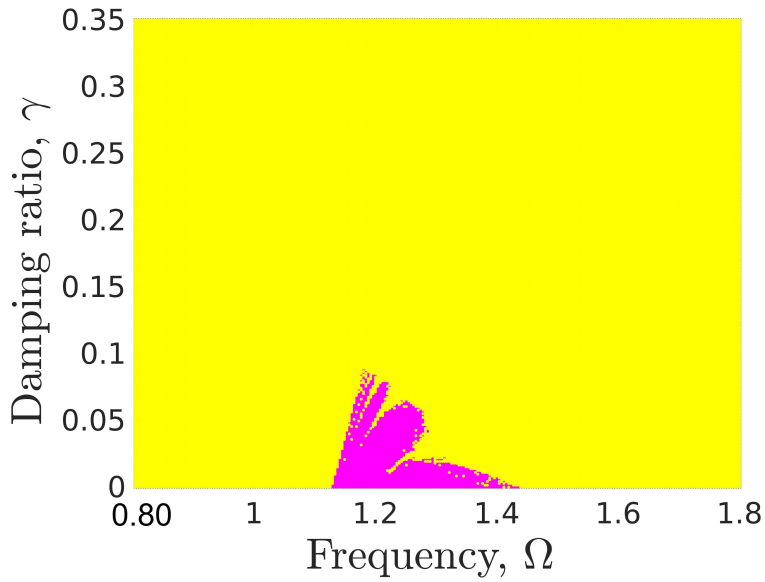


(ae) $\gamma = 0.30$

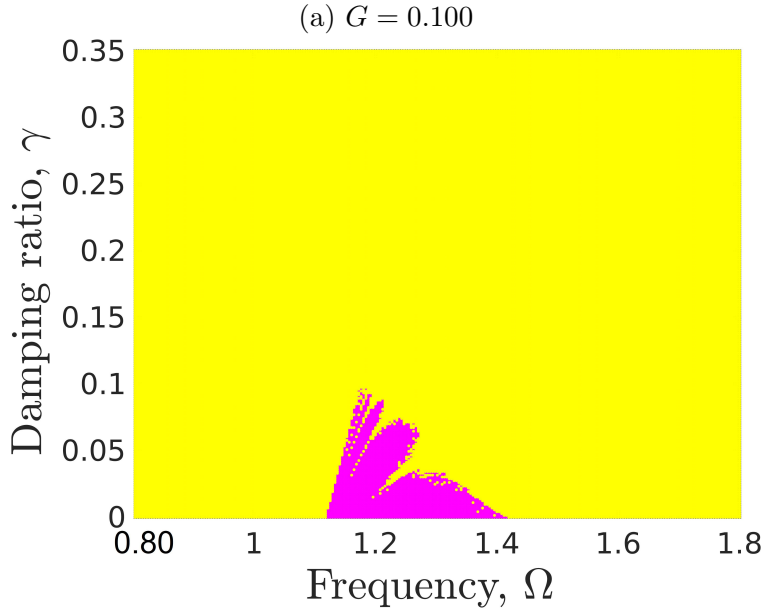
Figure S8: Two-colored forcing amplitude-frequency parameter space

S7 Two-Colored Damping Ratio-Forcing Frequency (γ vs. Ω) Parameter Space

31 two-colored damping ratio-forcing frequency parameter space for G ranges from $0.100 \sim 0.250$ with the increment of 0.05.

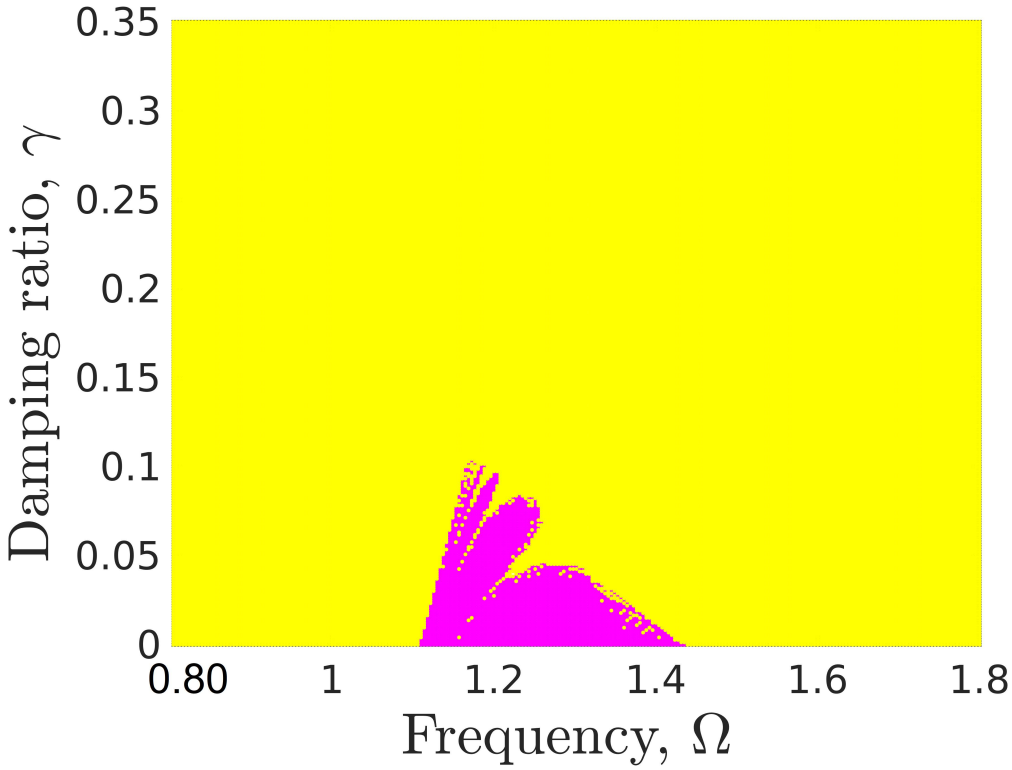


(a) $G = 0.100$

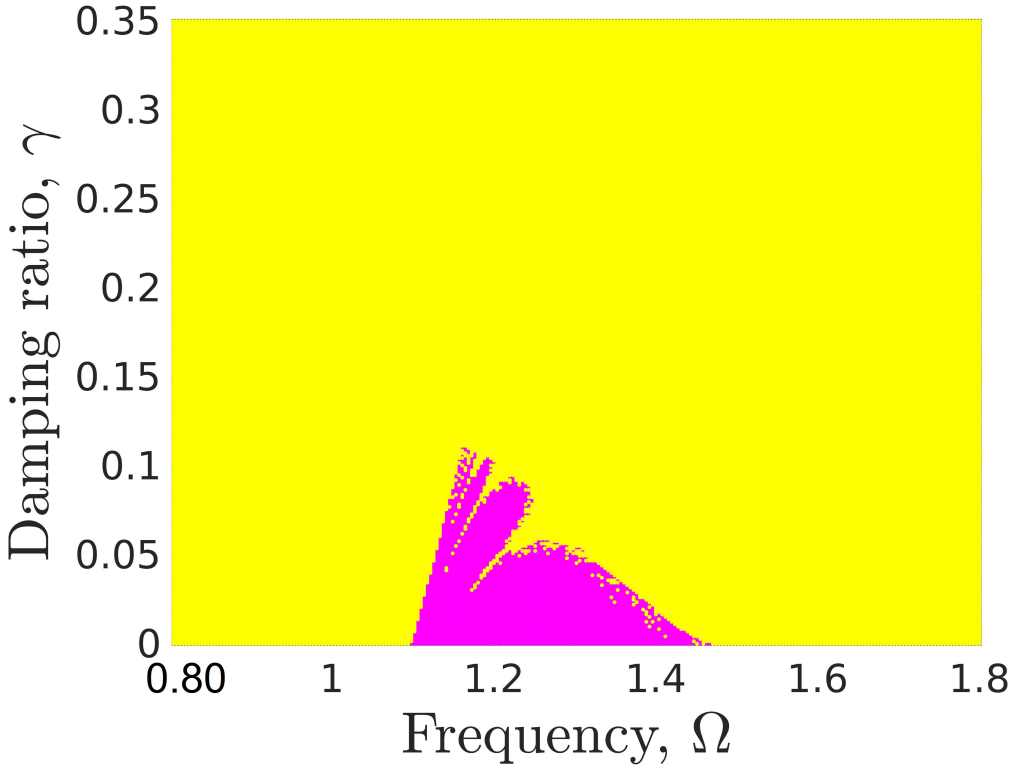


(b) $G = 0.105$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

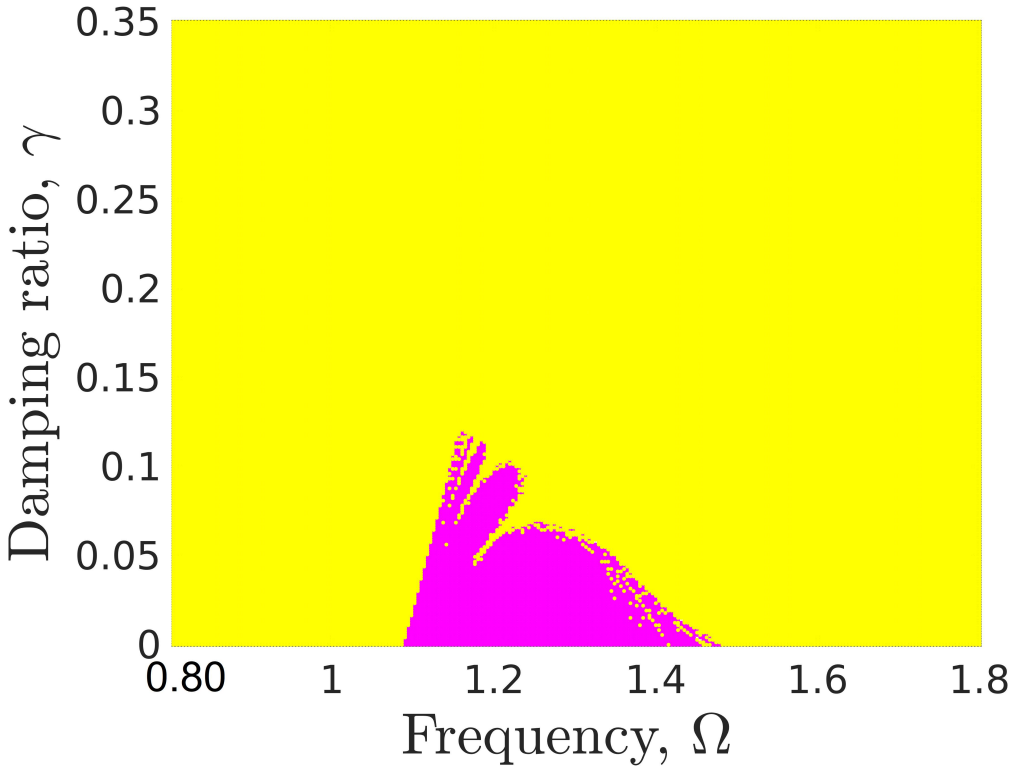


(c) $G = 0.110$

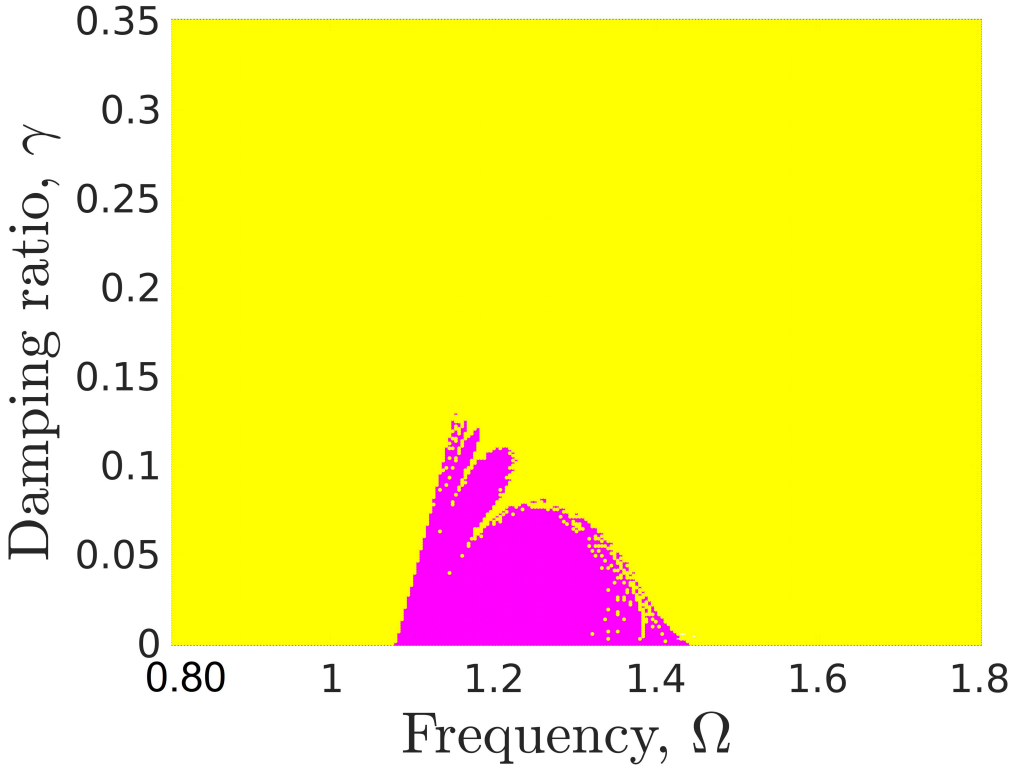


(d) $G = 0.115$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

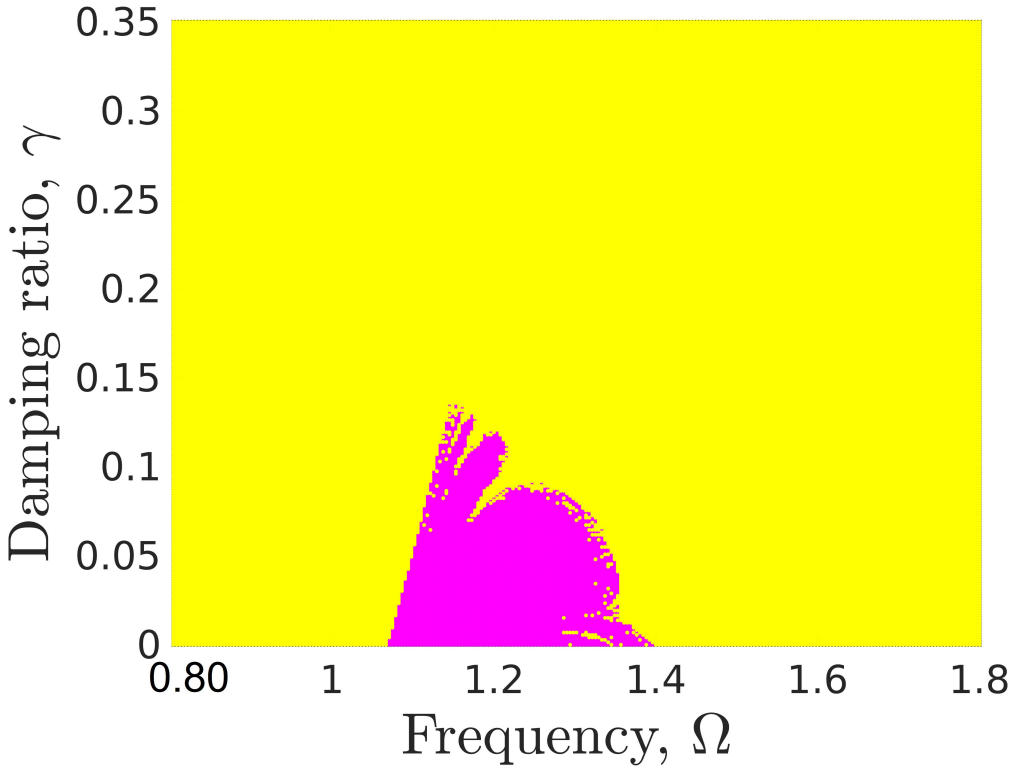


(e) $G = 0.120$

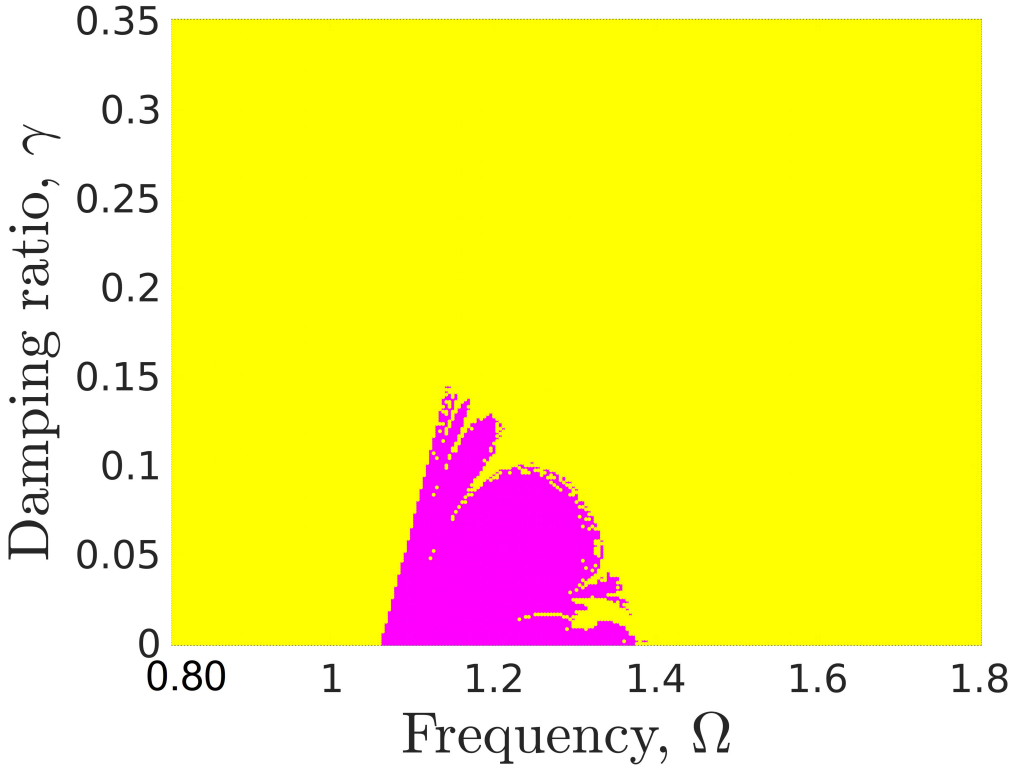


(f) $G = 0.125$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

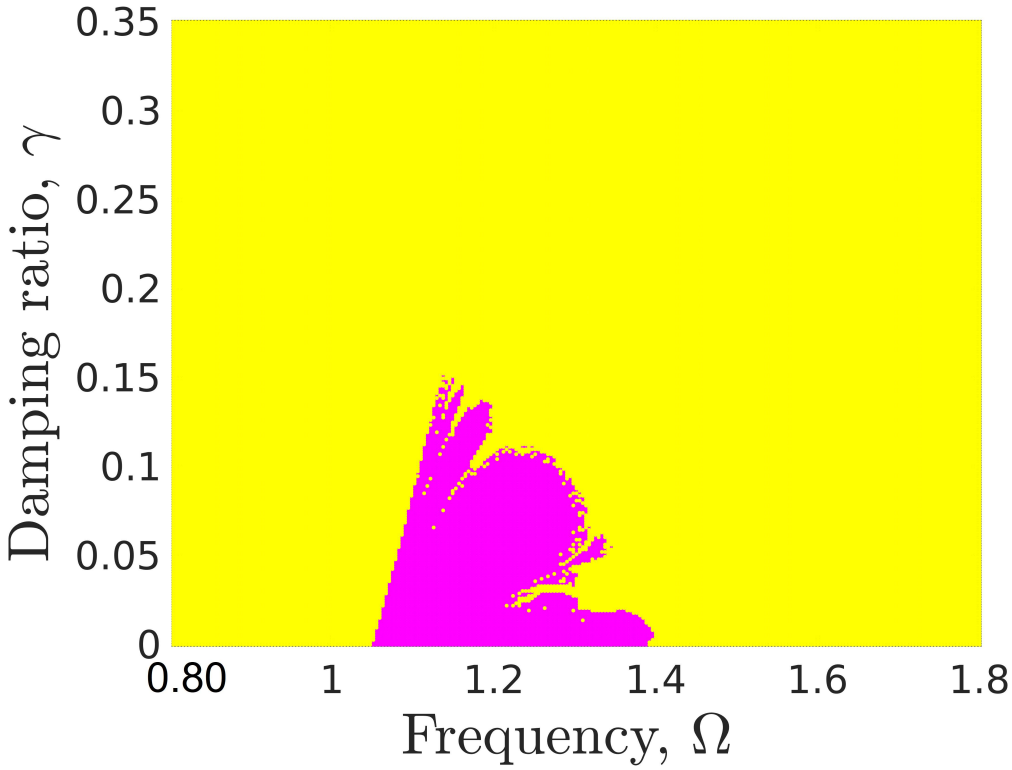


(g) $G = 0.130$

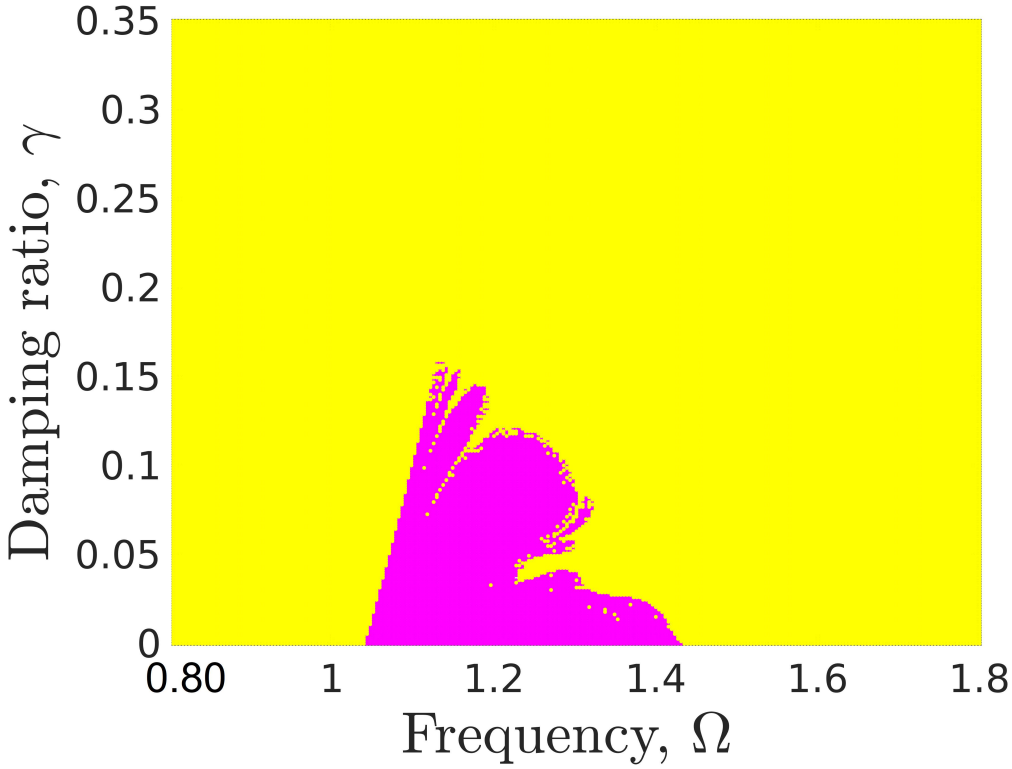


(h) $G = 0.135$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

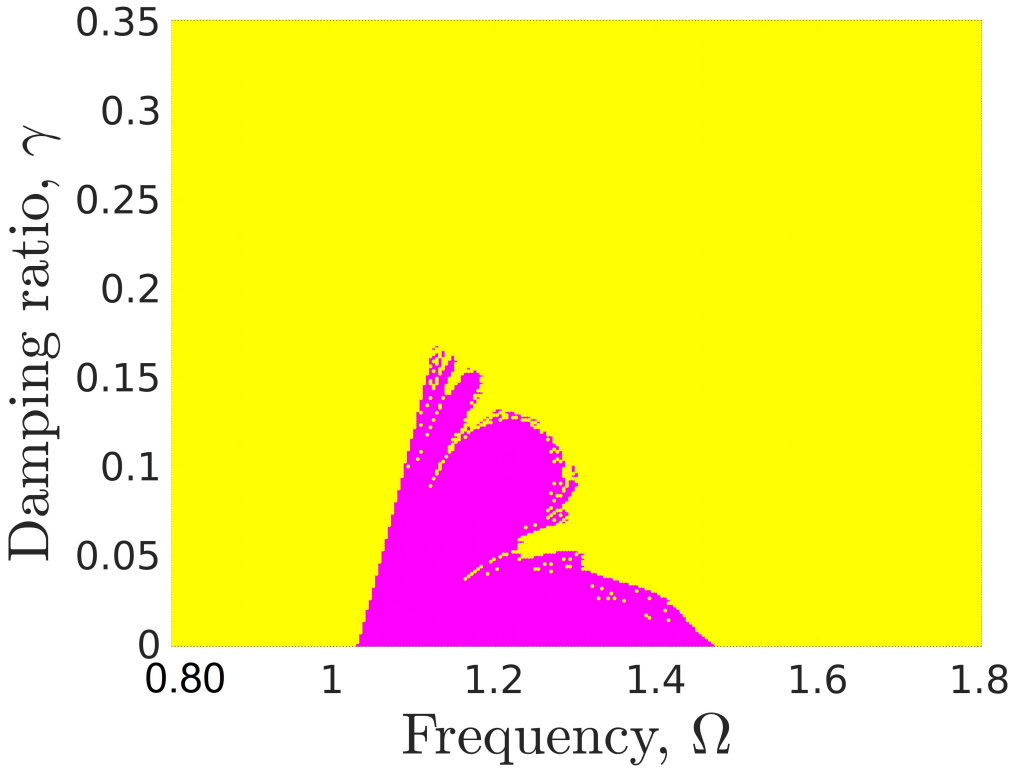


(i) $G = 0.140$

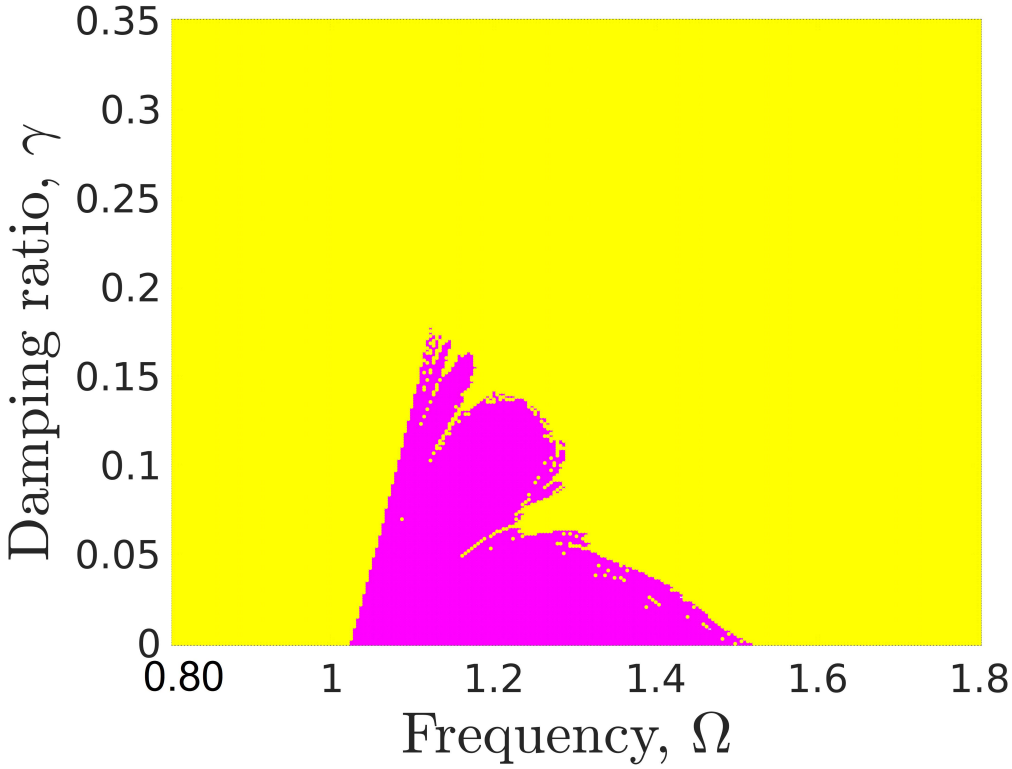


(j) $G = 0.145$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

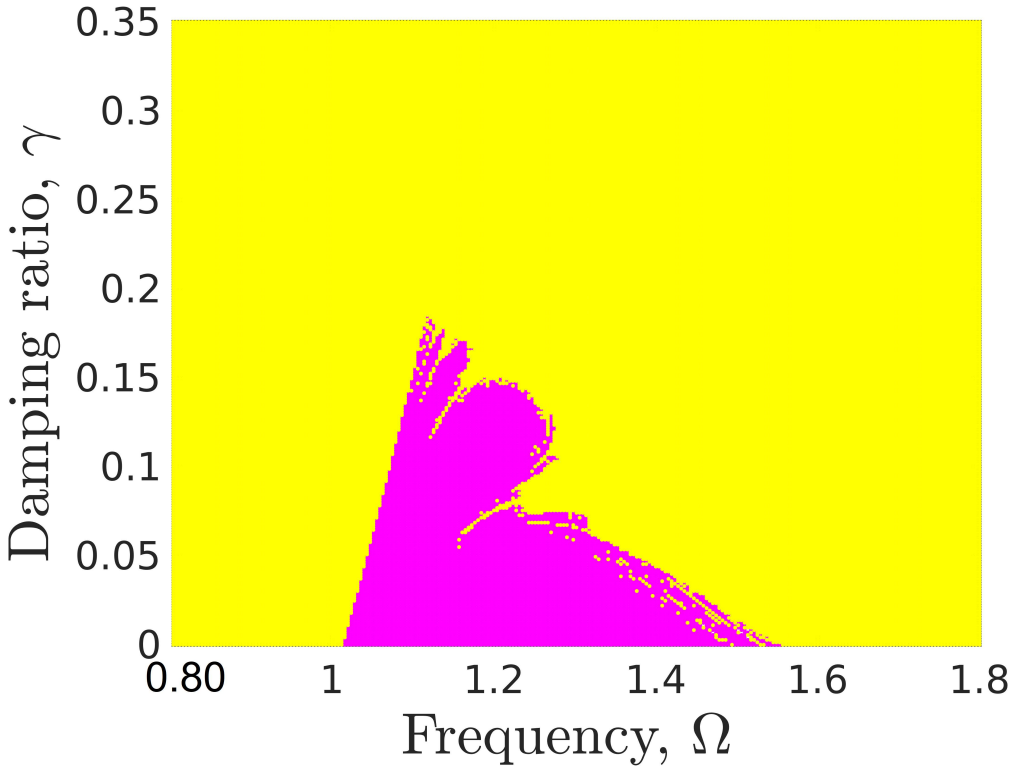


(k) $G = 0.150$

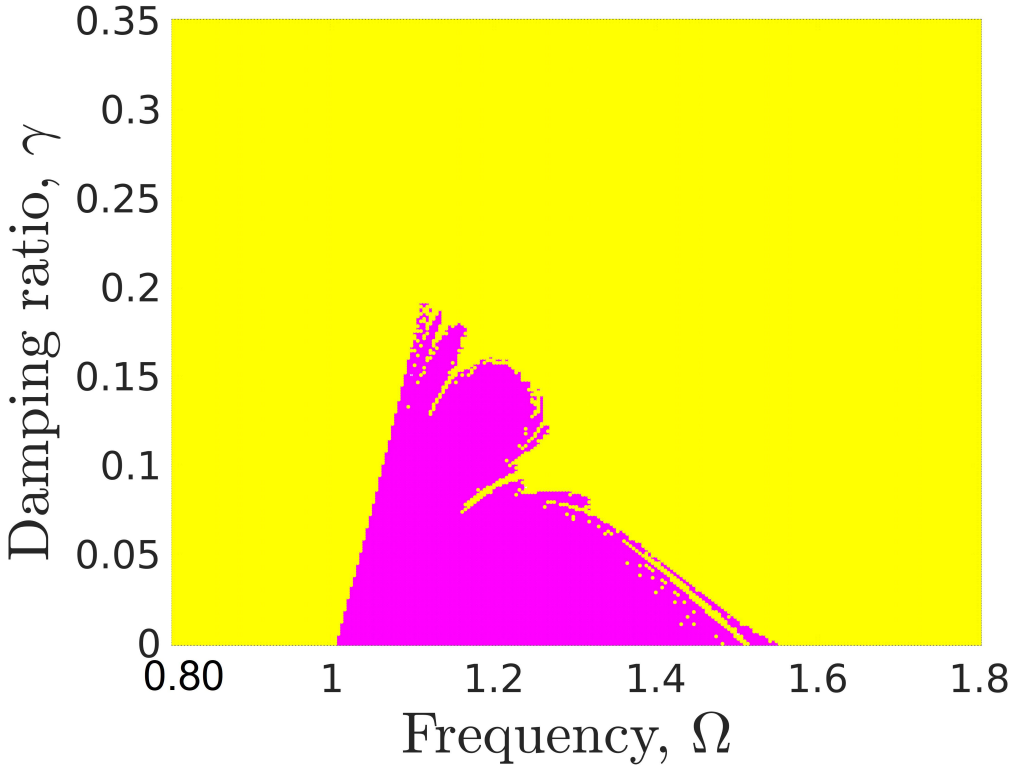


(l) $G = 0.155$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

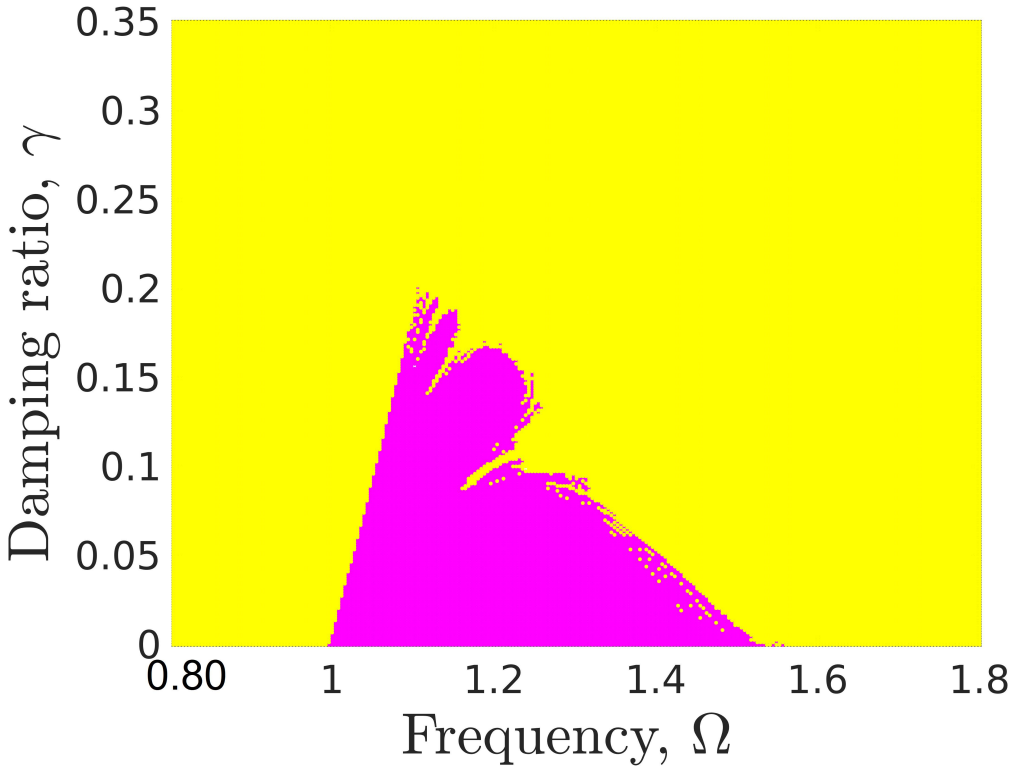


(m) $G = 0.160$

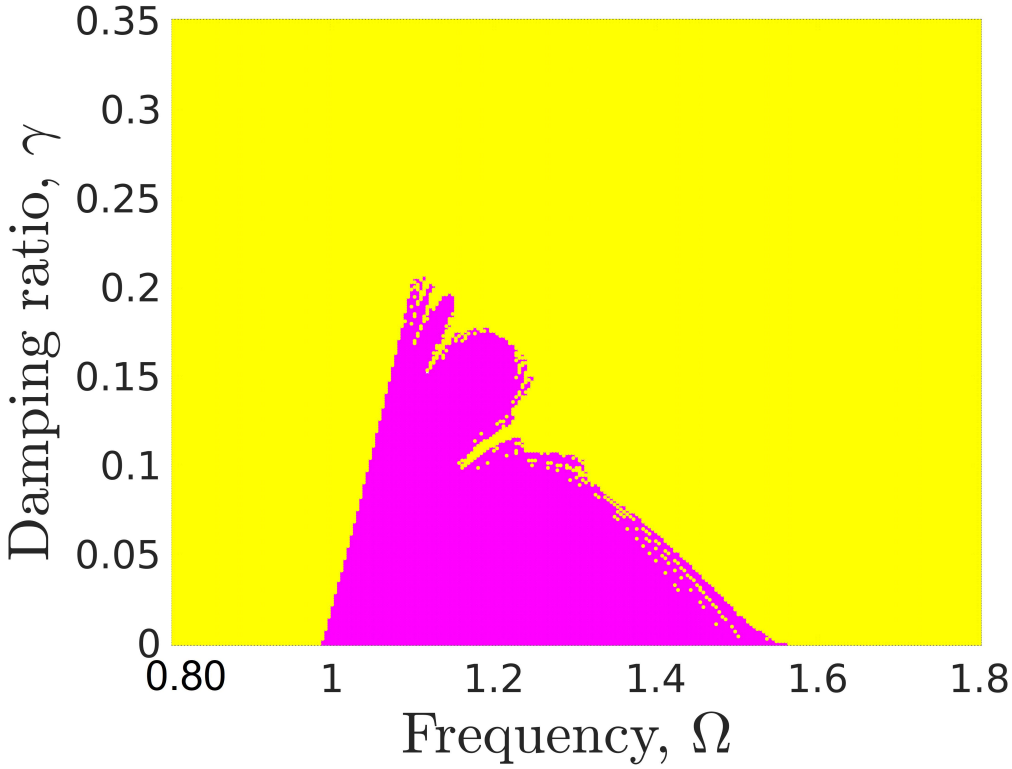


(n) $G = 0.165$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

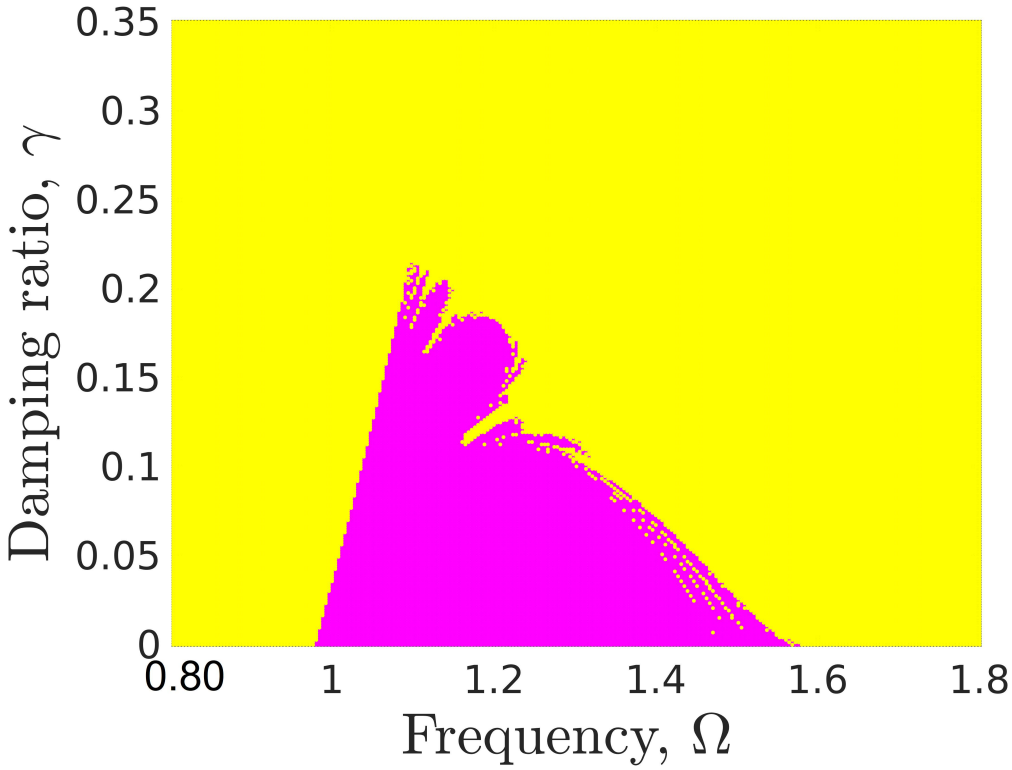


(o) $G = 0.170$

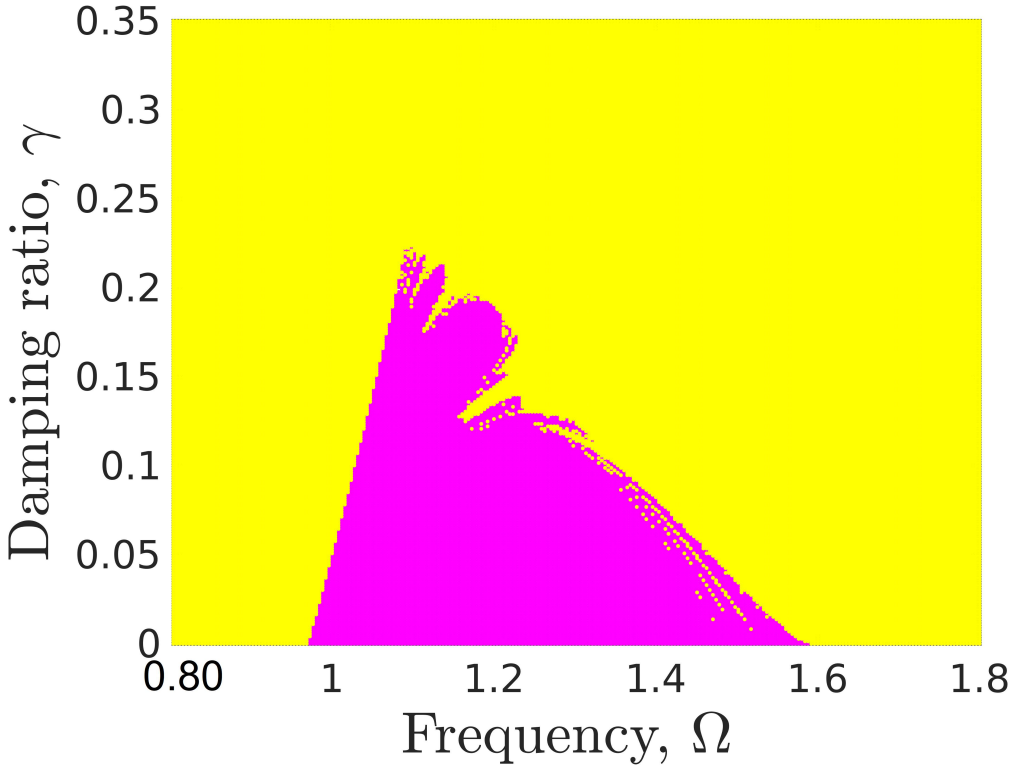


(p) $G = 0.175$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

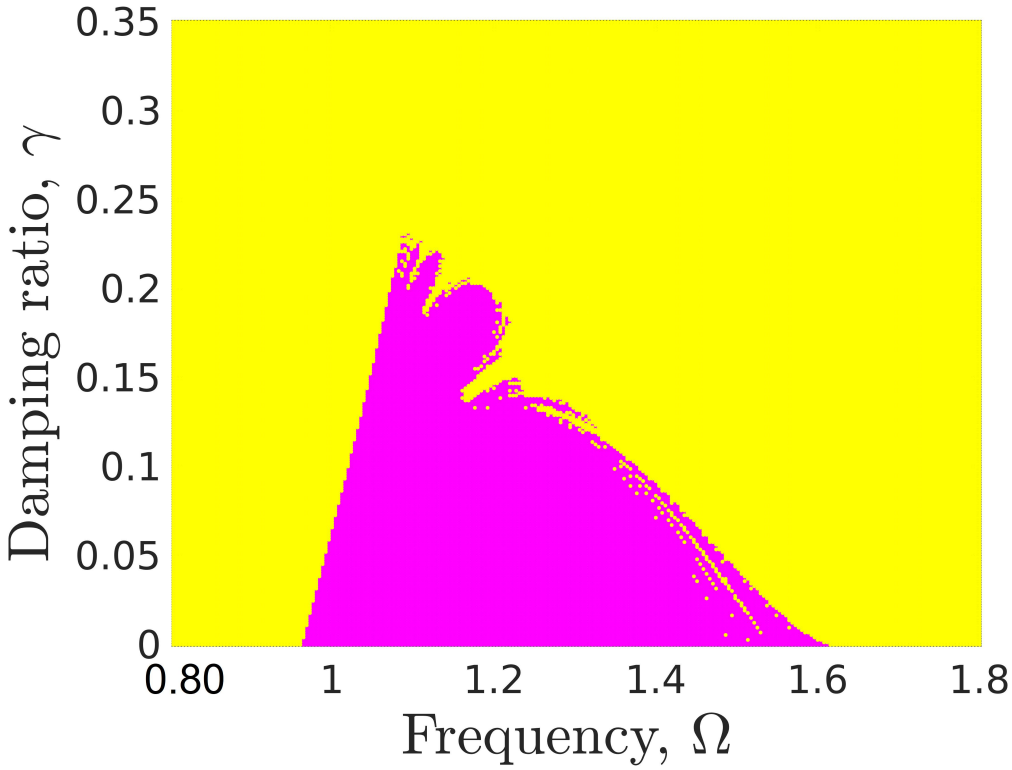


(q) $G = 0.180$

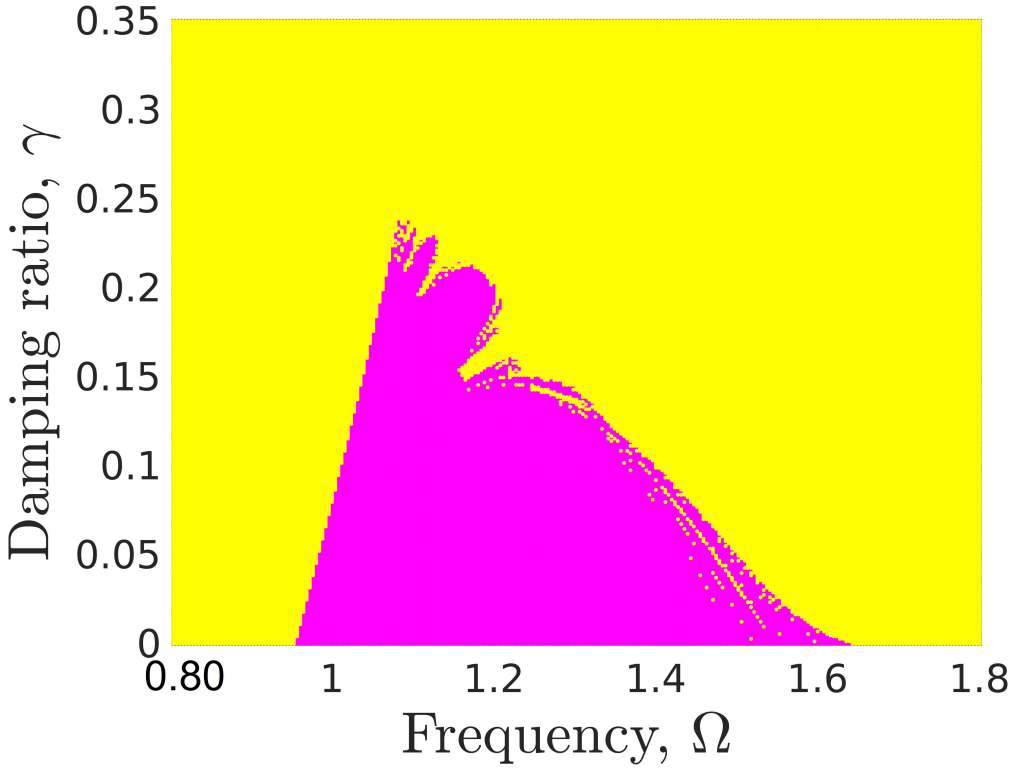


(r) $G = 0.185$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

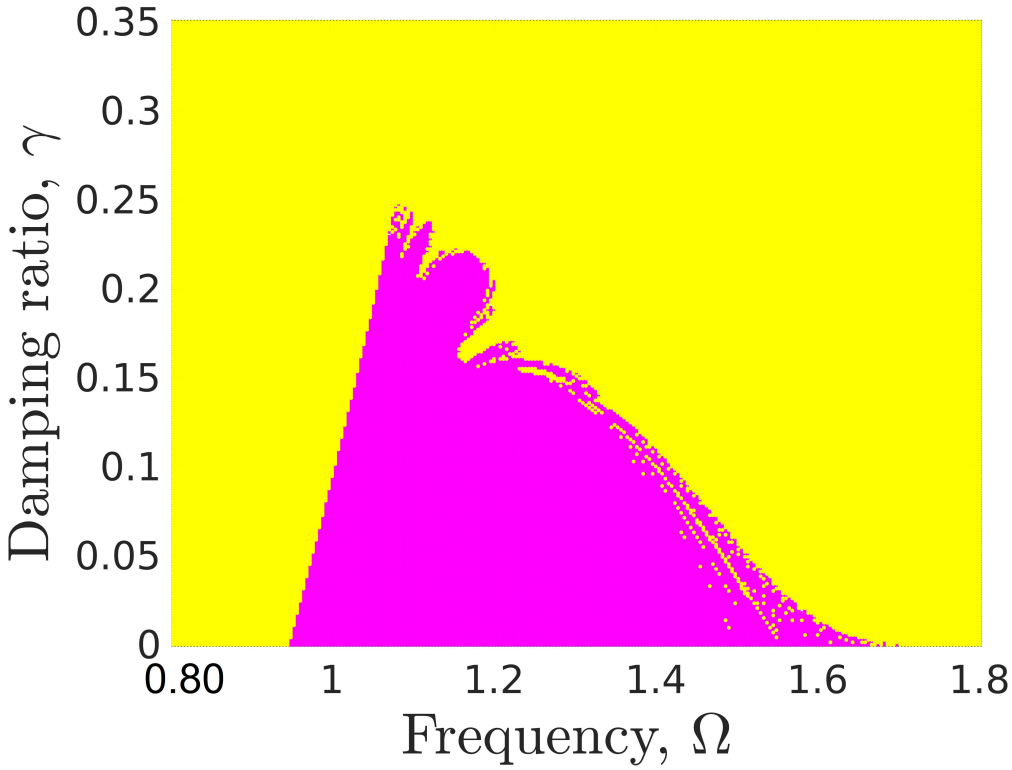


(s) $G = 0.190$

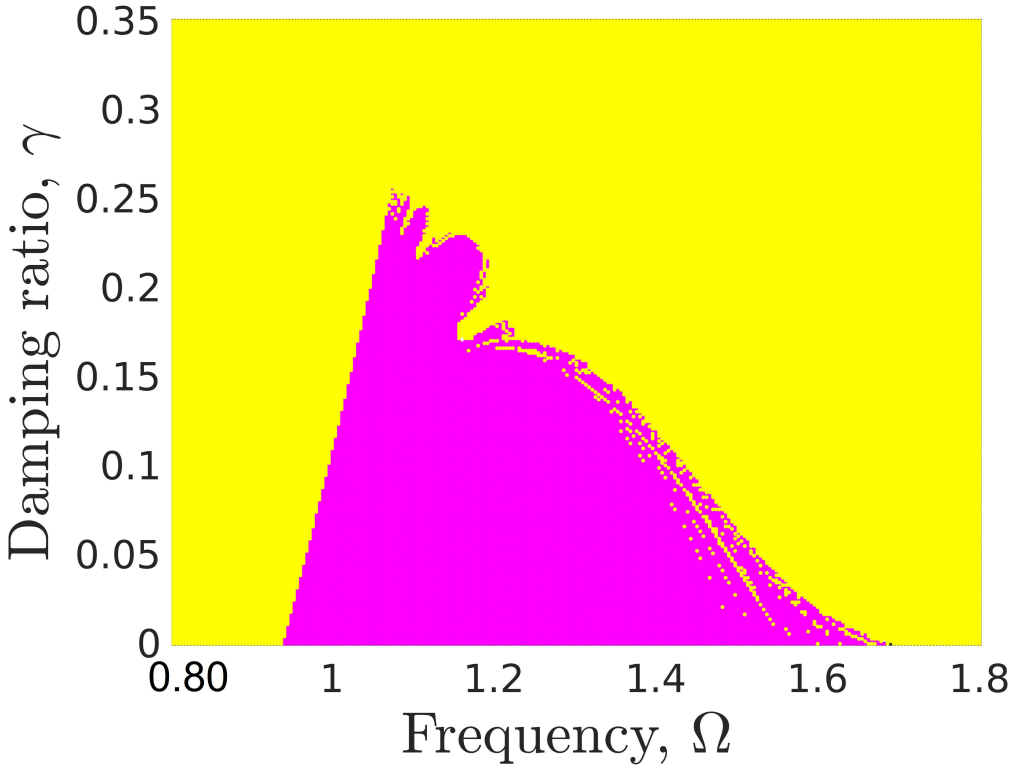


(t) $G = 0.195$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

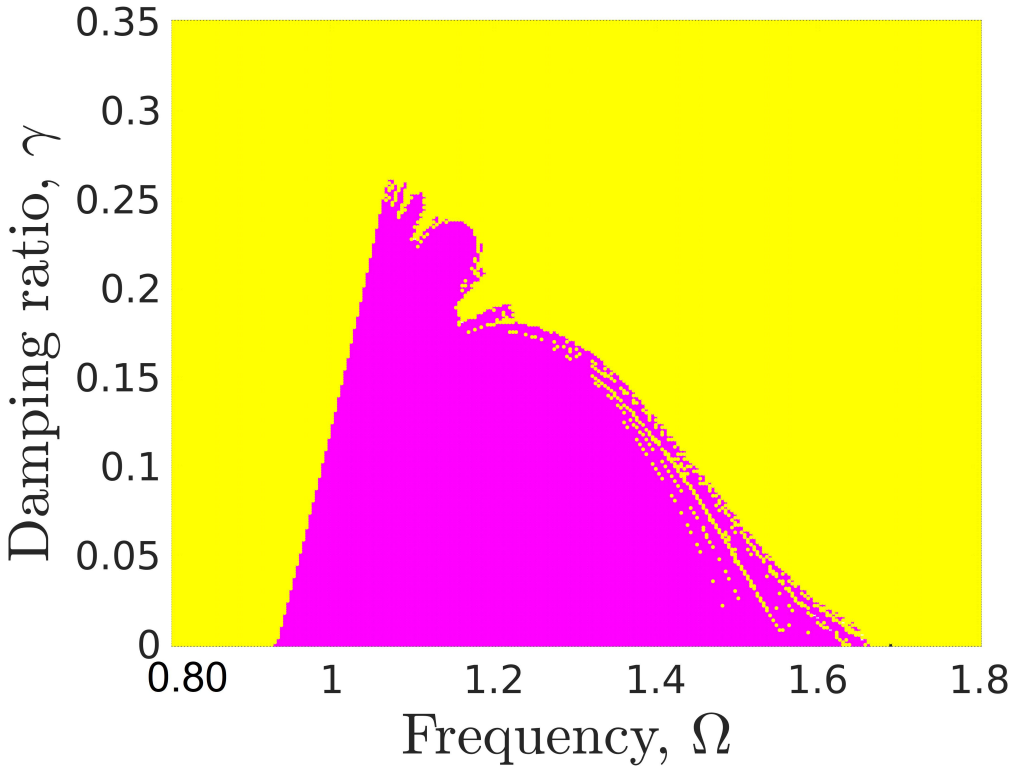


(u) $G = 0.200$

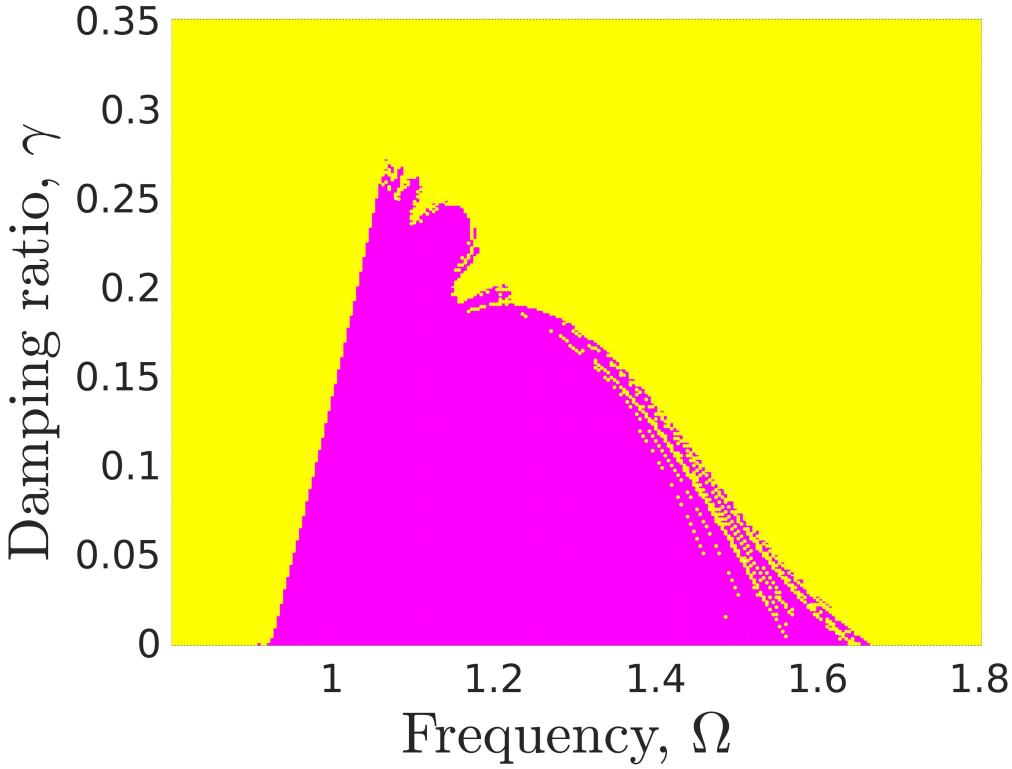


(v) $G = 0.205$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

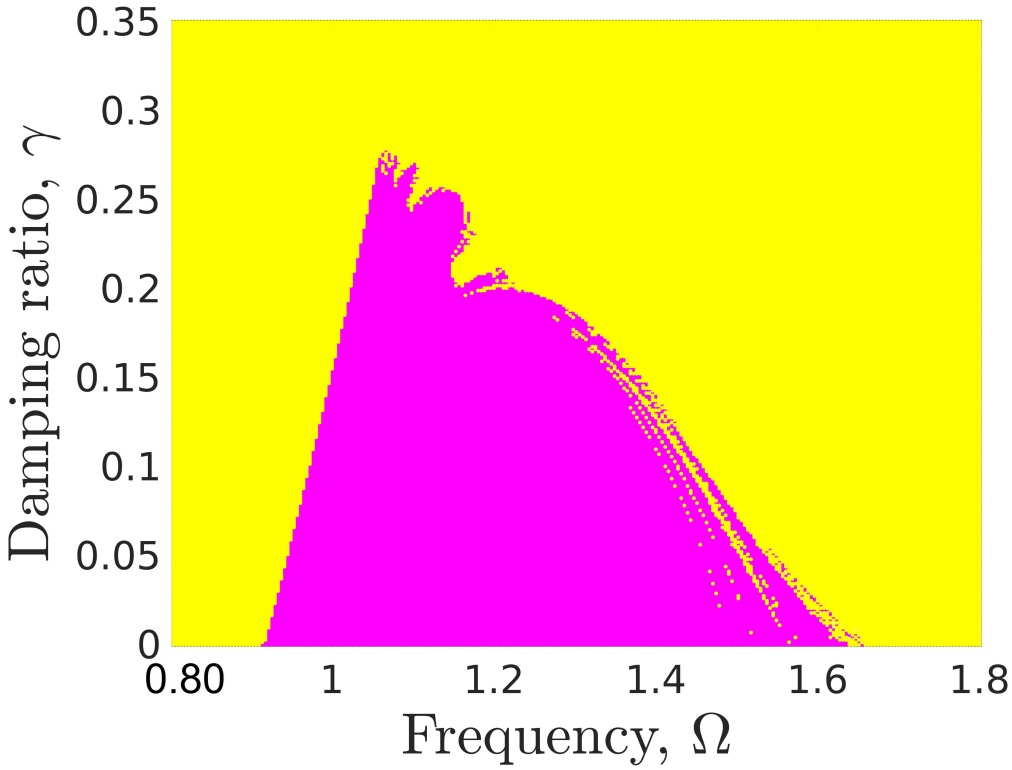


(w) $G = 0.210$

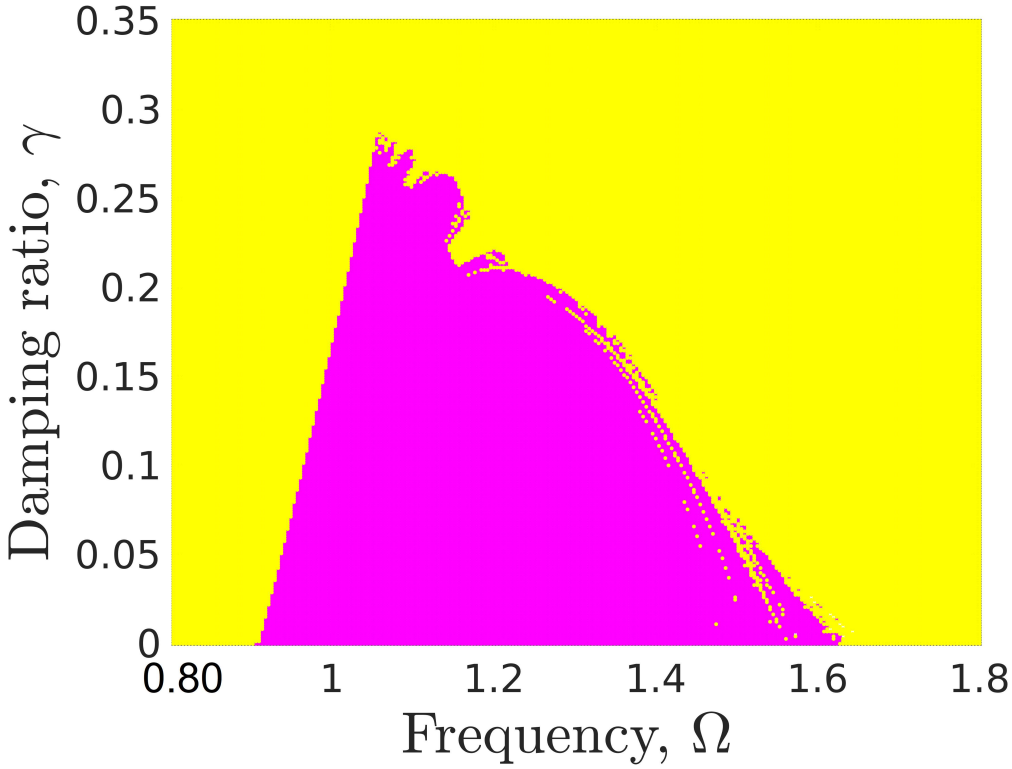


(x) $G = 0.215$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

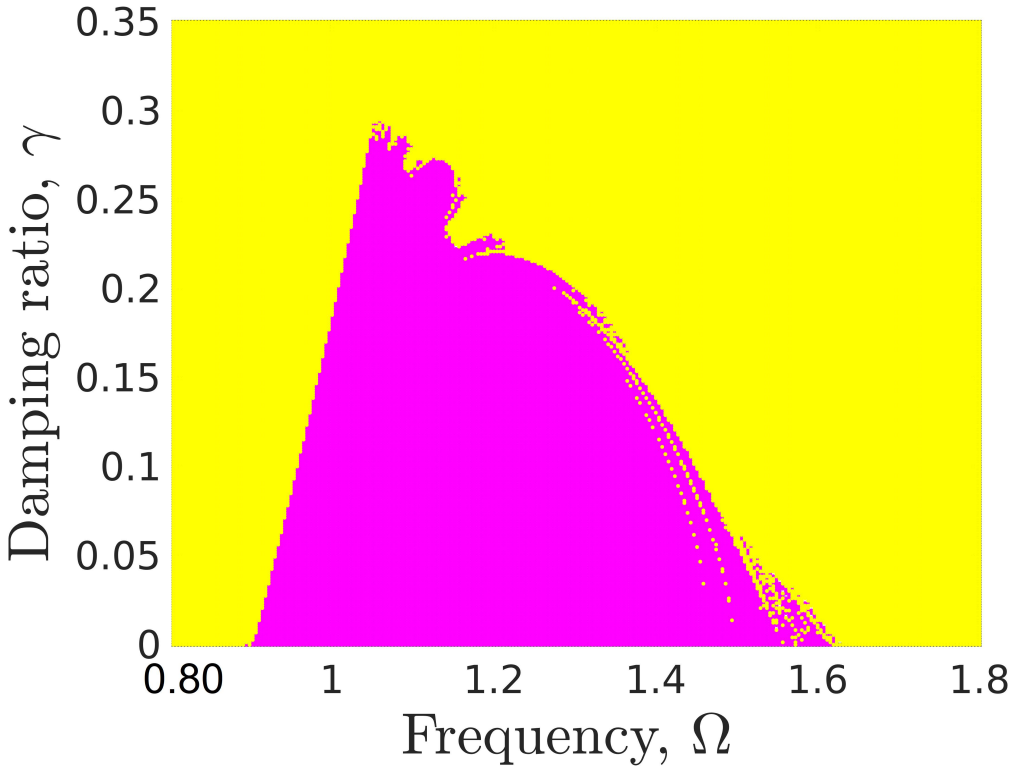


(y) $G = 0.220$

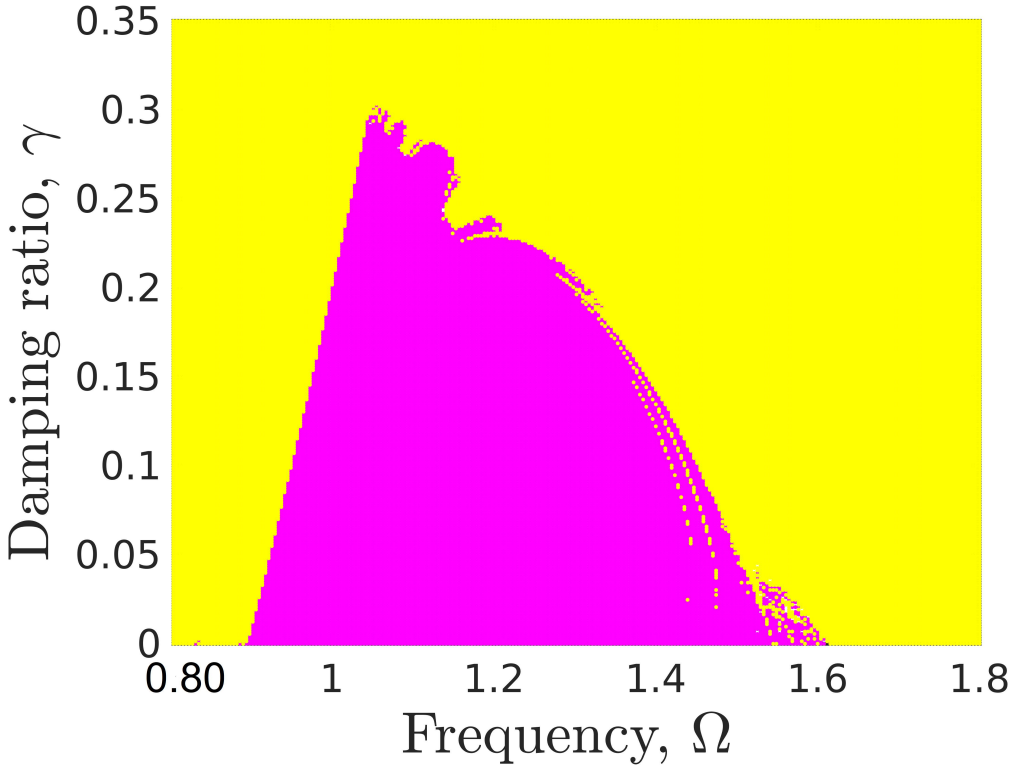


(z) $G = 0.225$

Figure S9: Two-colored damping ratio-forcing frequency parameter space

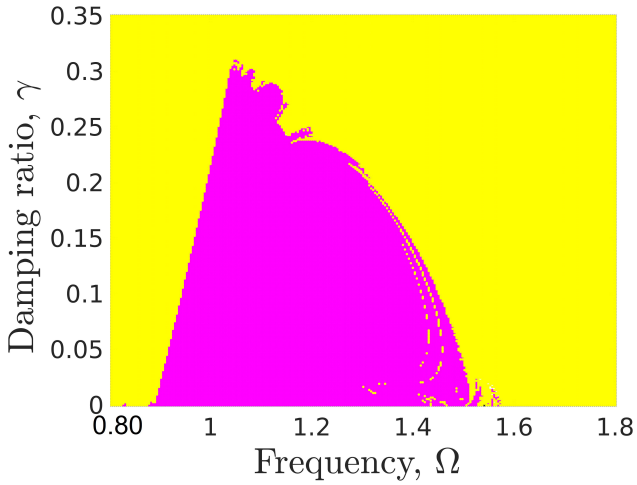


(aa) $G = 0.230$

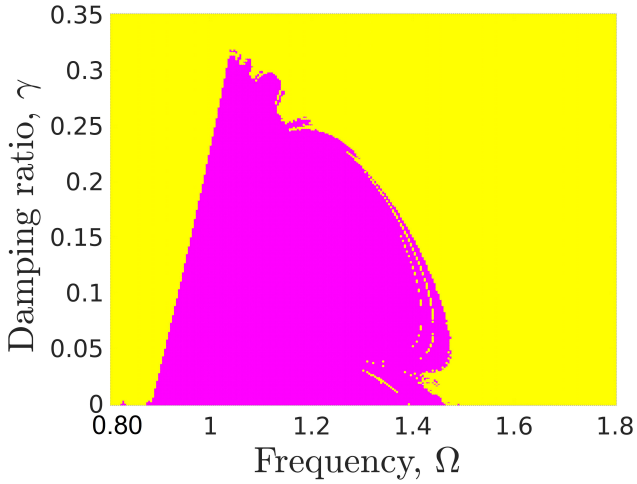


(ab) $G = 0.235$

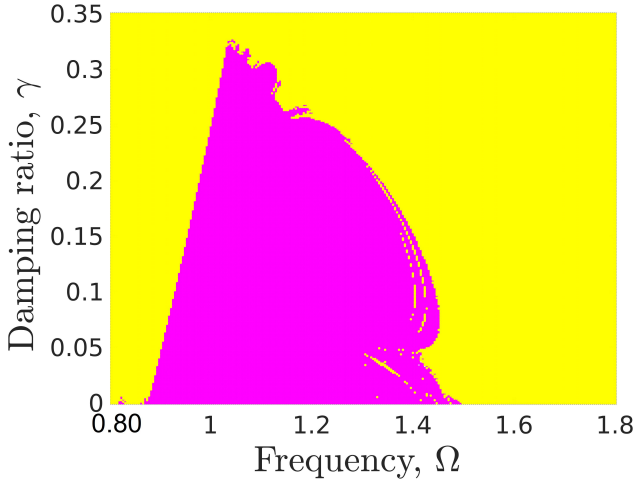
Figure S9: Two-colored damping ratio-forcing frequency parameter space



(ac) $G = 0.240$



(ad) $G = 0.245$



(ae) $G = 0.250$

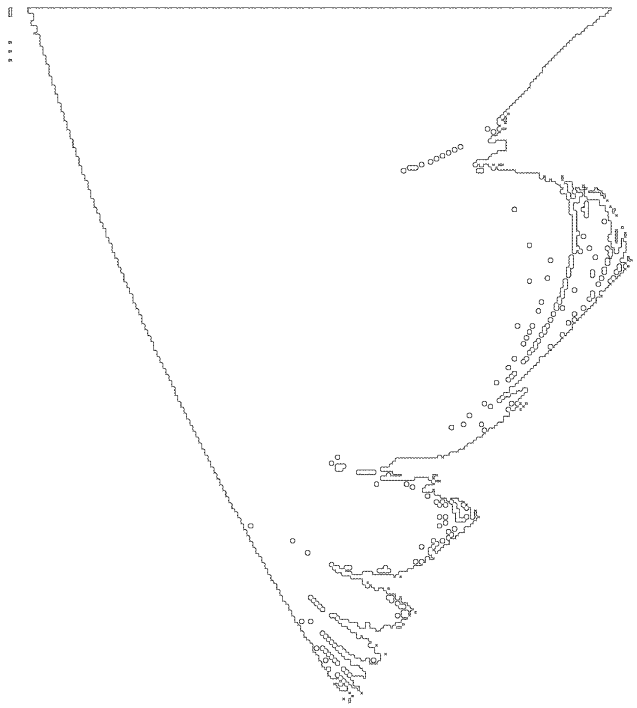
Figure S9: Two-colored damping ratio-forcing frequency parameter space

S8 Fractal Boundary between Inter-well and Intra-well Motion for Forcing Amplitude-Frequency (G vs. Ω) Parameter Space

31 two-colored forcing amplitude-frequency parameter space fractal boundary between inter-well and intra-well motion for γ ranges from $0.001 \sim 0.30$ with the increment of 0.01,

(a) $\gamma = 0.001$ (b) $\gamma = 0.01$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)



(c) $\gamma = 0.02$

(d) $\gamma = 0.03$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

(e) $\gamma = 0.04$

(f) $\gamma = 0.05$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

(g) $\gamma = 0.06$

(h) $\gamma = 0.07$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

(i) $\gamma = 0.08$

(j) $\gamma = 0.09$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

(k) $\gamma = 0.10$

(l) $\gamma = 0.11$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

(m) $\gamma = 0.12$

(n) $\gamma = 0.13$

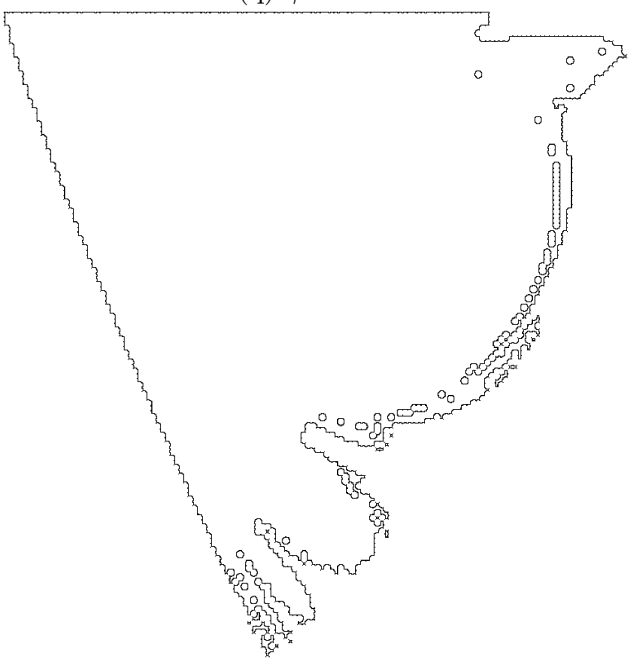
Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

(o) $\gamma = 0.14$

(p) $\gamma = 0.15$

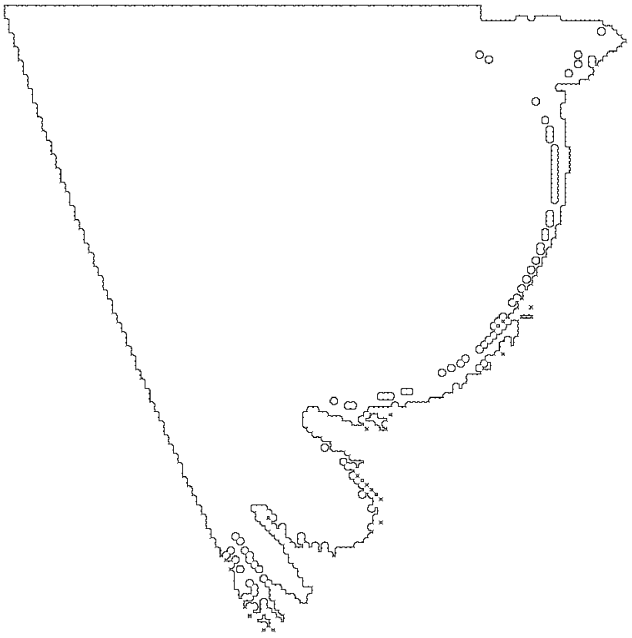
Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

(q) $\gamma = 0.16$

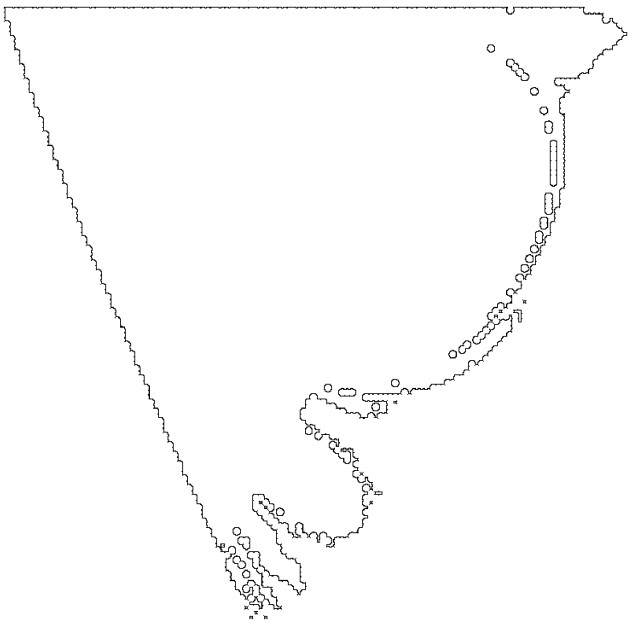


(r) $\gamma = 0.17$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

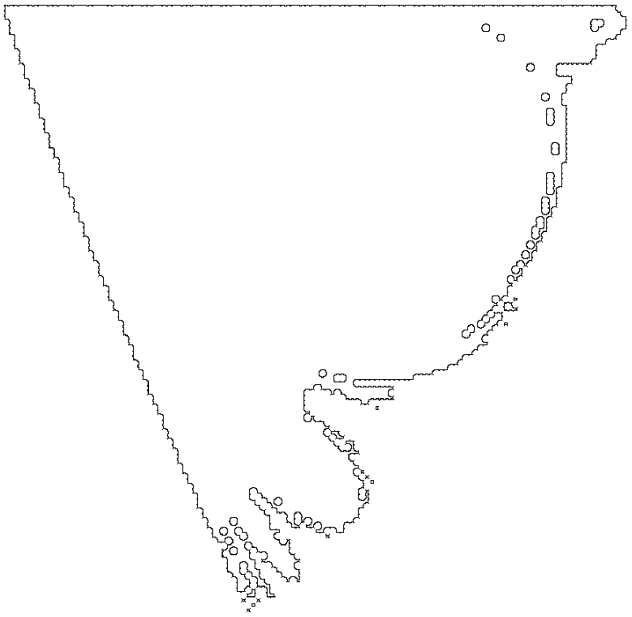


(s) $\gamma = 0.18$



(t) $\gamma = 0.19$

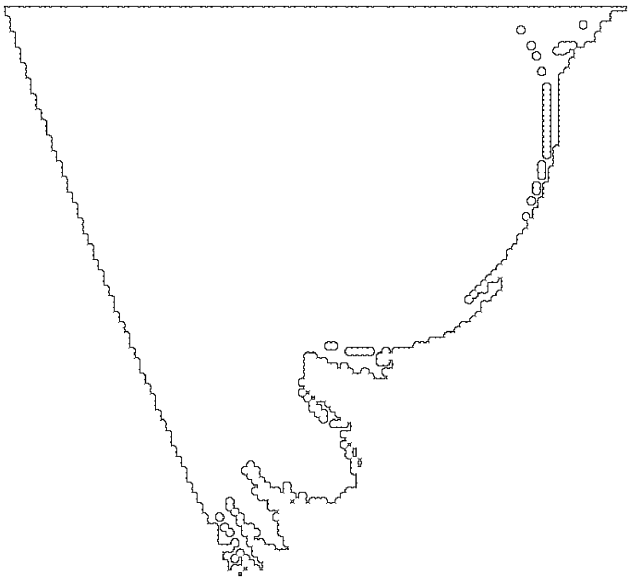
Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)



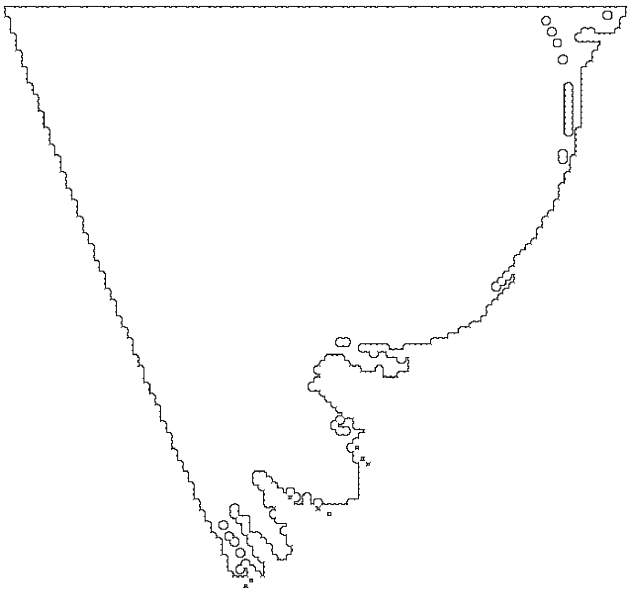
(u) $\gamma = 0.20$

(v) $\gamma = 0.21$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

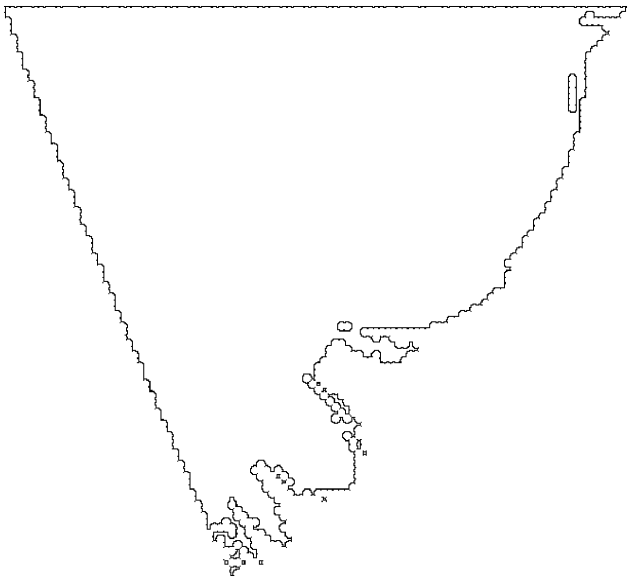


(w) $\gamma = 0.22$

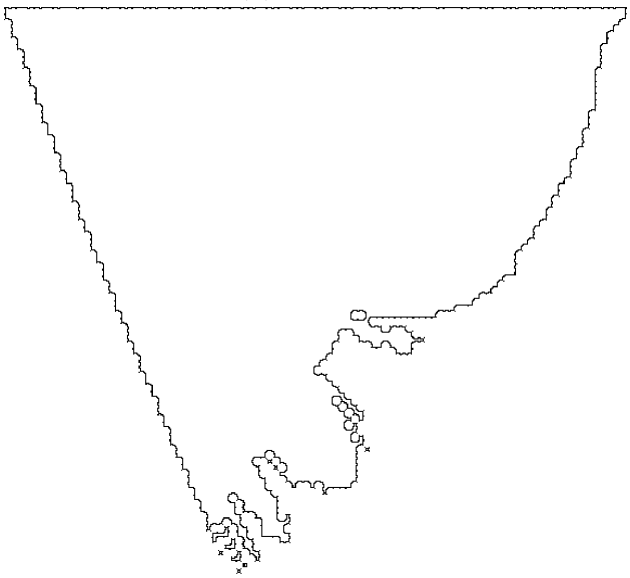


(x) $\gamma = 0.23$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

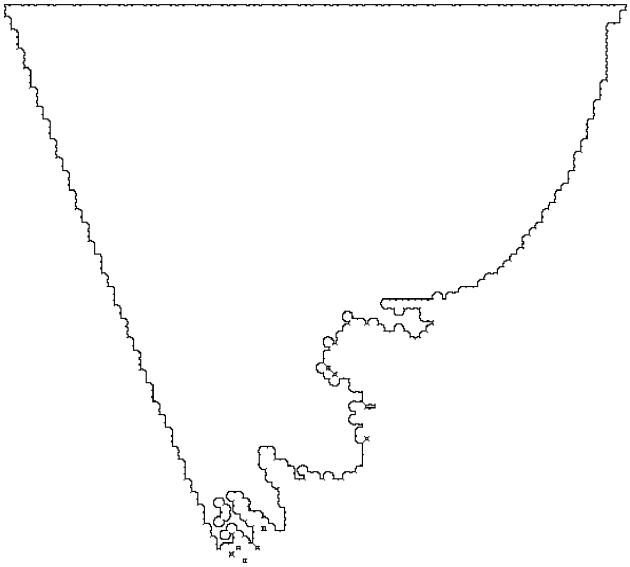


(y) $\gamma = 0.24$

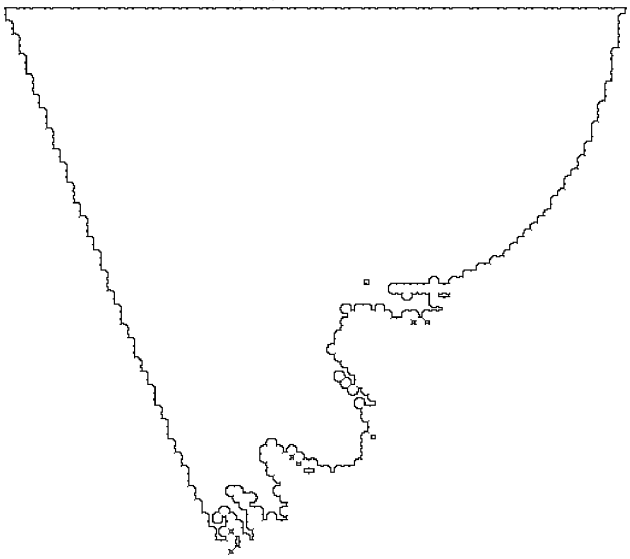


(z) $\gamma = 0.25$

Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

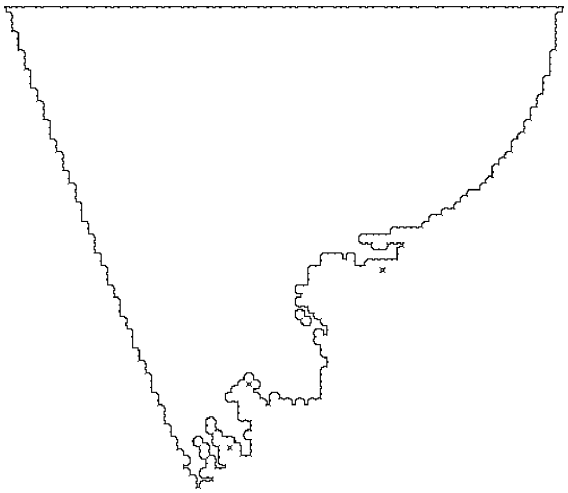


(aa) $\gamma = 0.26$

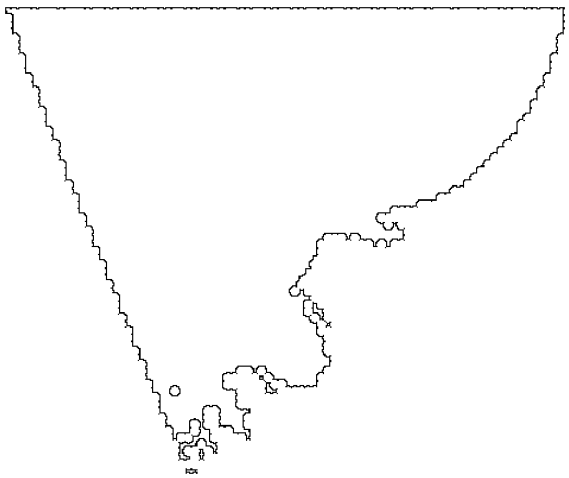


(ab) $\gamma = 0.27$

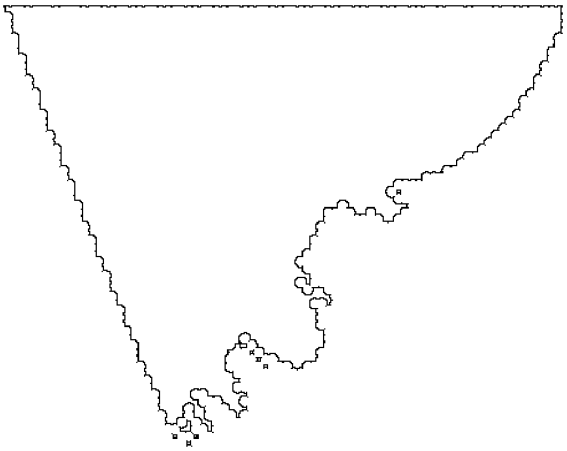
Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)



(ac) $\gamma = 0.28$



(ad) $\gamma = 0.29$

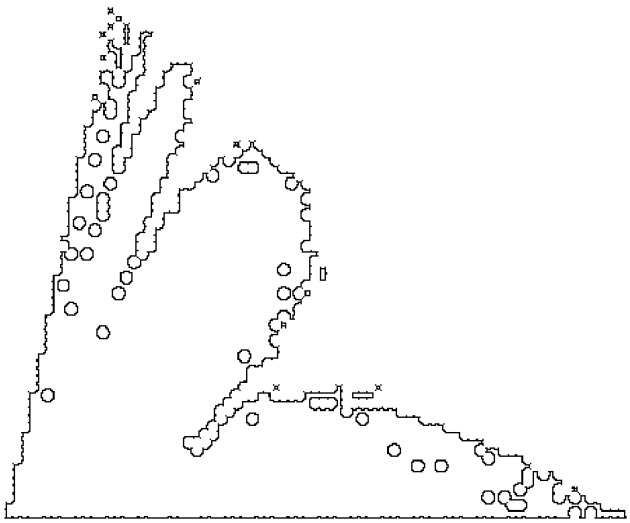


(ae) $\gamma = 0.30$

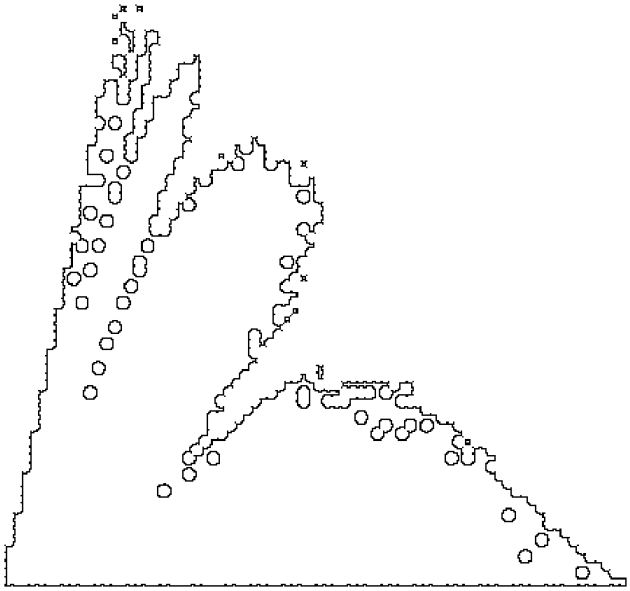
Figure S10: Fractal boundary between inter-well and intra-well motion (two-colored forcing amplitude-frequency parameter space)

S9 Two-Colored Damping Ratio-Forcing Frequency (γ vs. Ω) Parameter Space Boundary between Inter-well and Intra-well Motion

31 two-colored damping ratio-forcing frequency parameter space fractal boundary between inter-well and intra-well motion for G ranges from $0.100 \sim 0.250$ with the increment of 0.05.

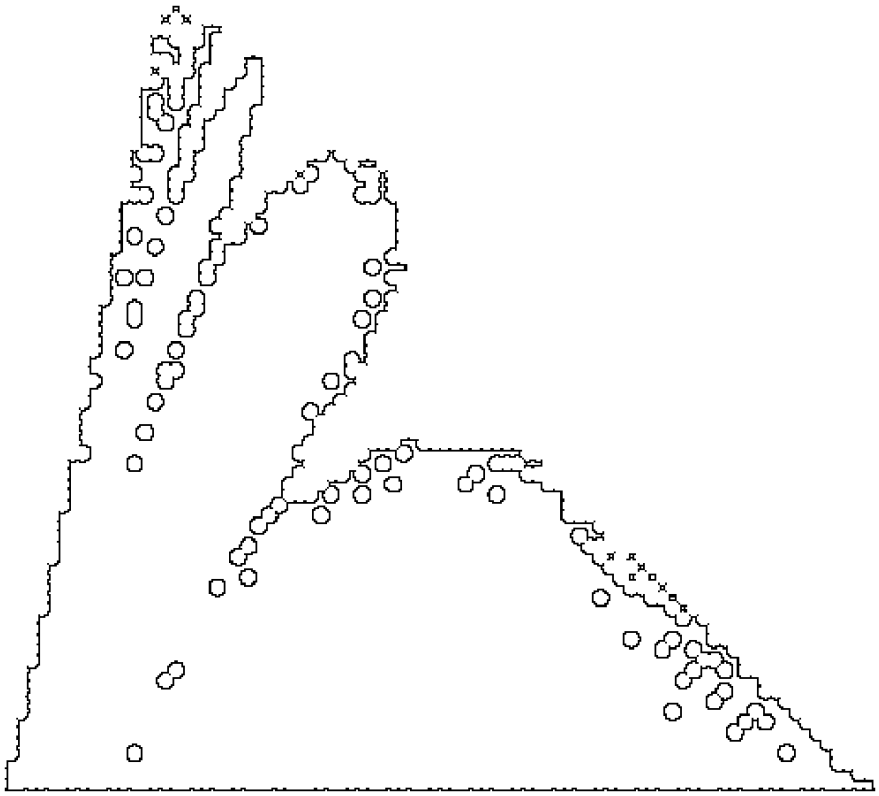


(a) $G = 0.100$

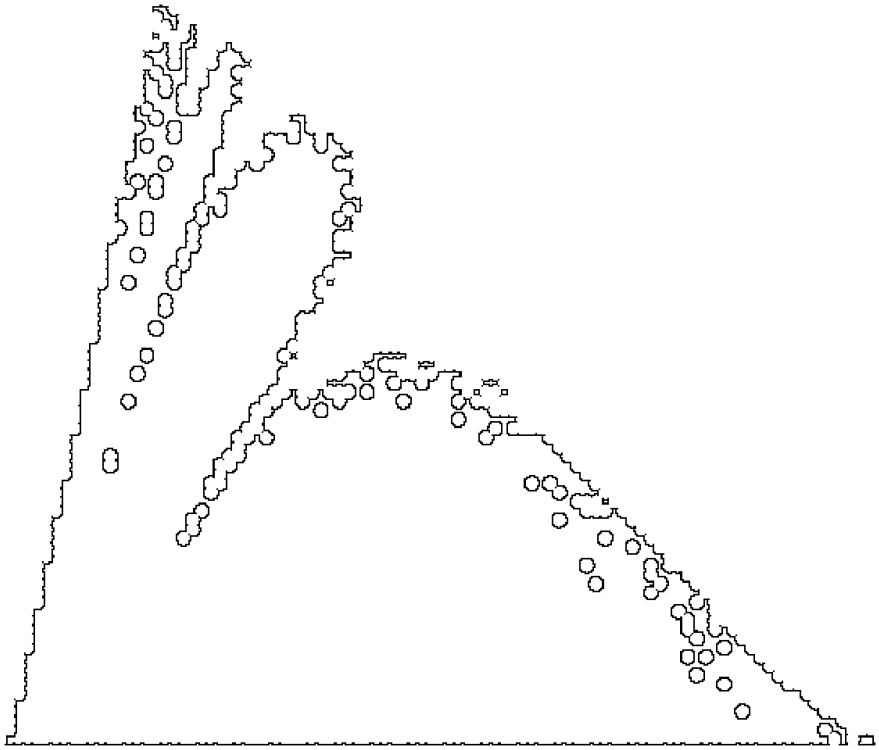


(b) $G = 0.105$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

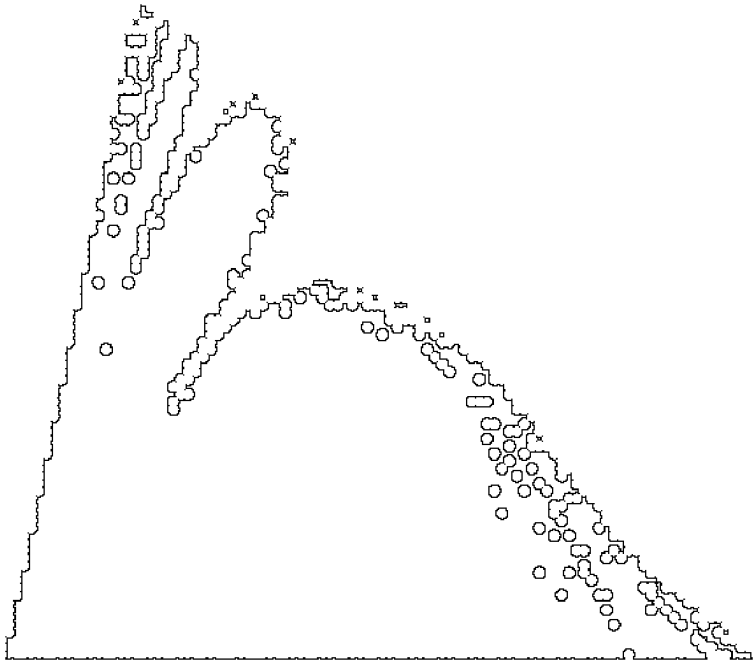


(c) $G = 0.110$

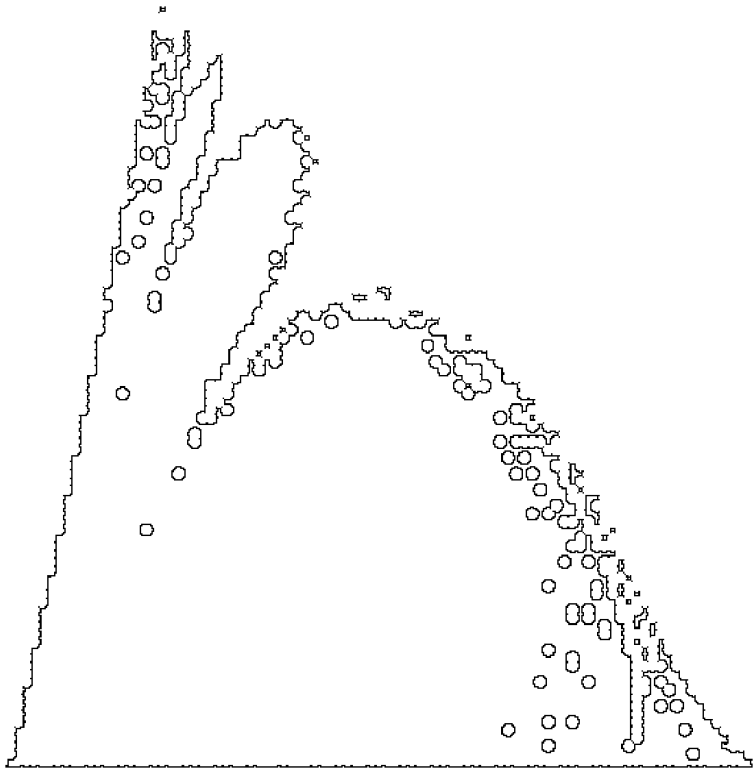


(d) $G = 0.115$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

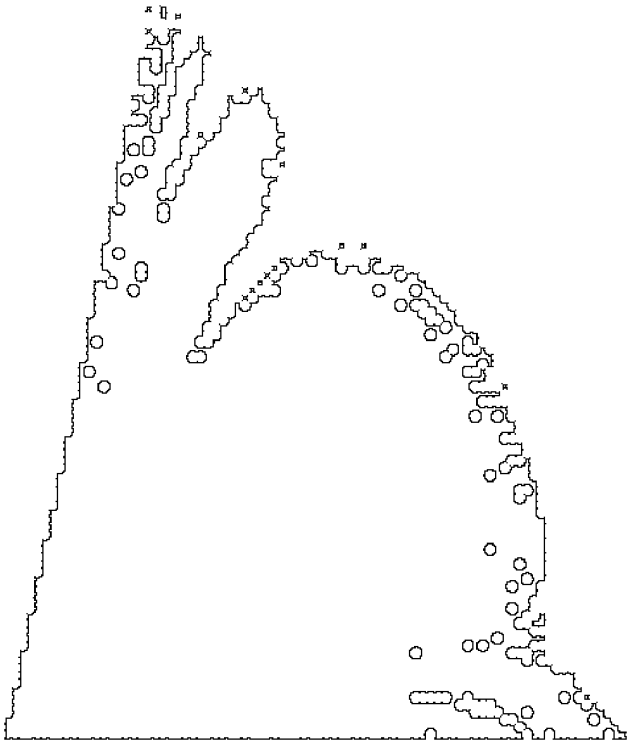


(e) $G = 0.120$

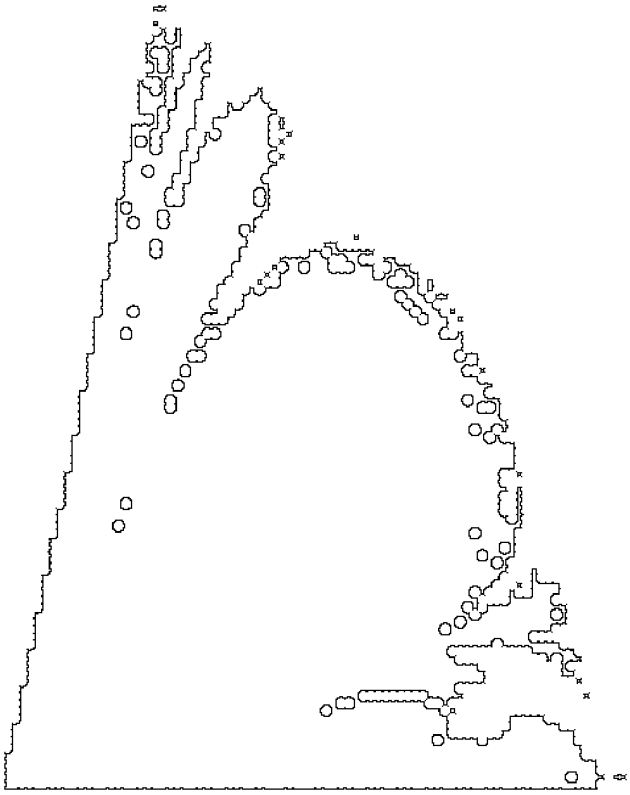


(f) $G = 0.125$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

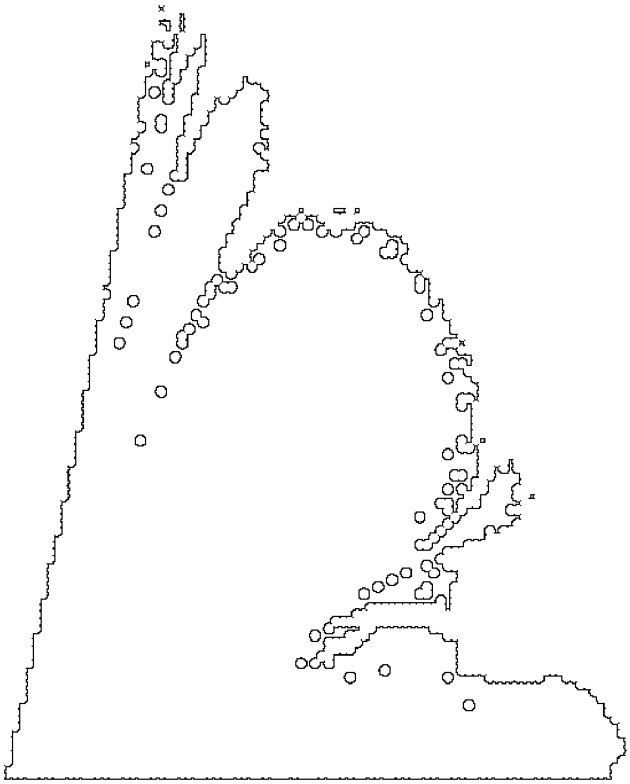


(g) $G = 0.130$



(h) $G = 0.135$

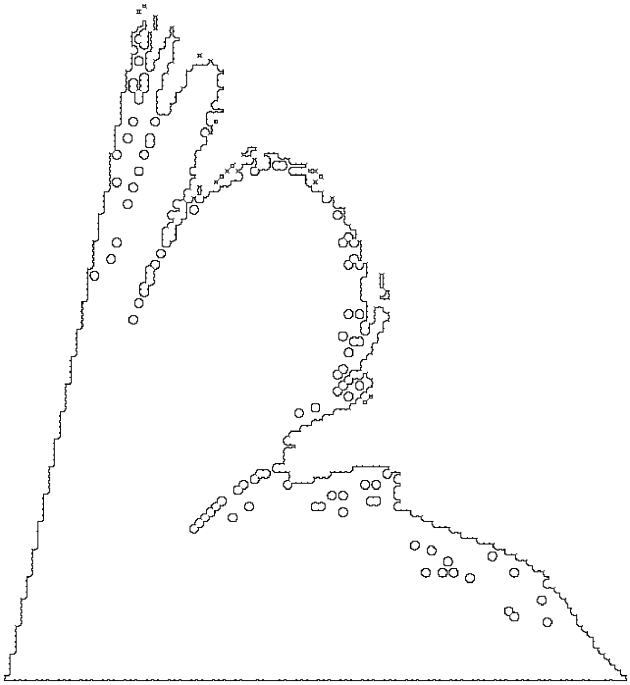
Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)



(i) $G = 0.140$

(j) $G = 0.145$

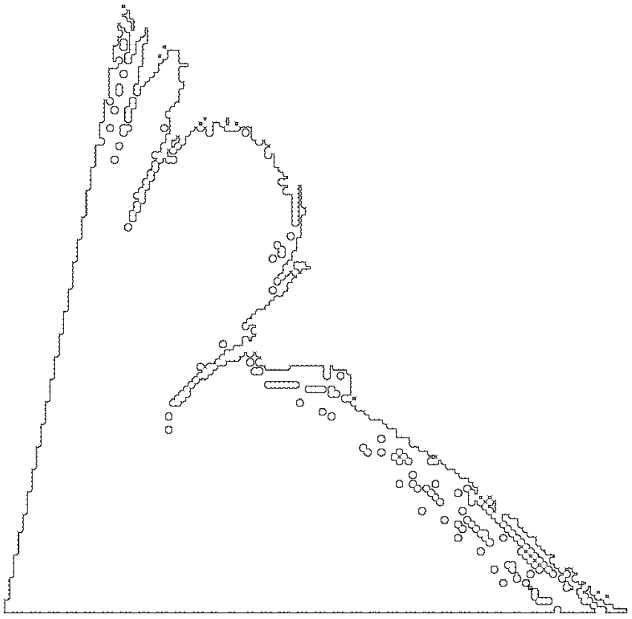
Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)



(k) $G = 0.150$

(l) $G = 0.155$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

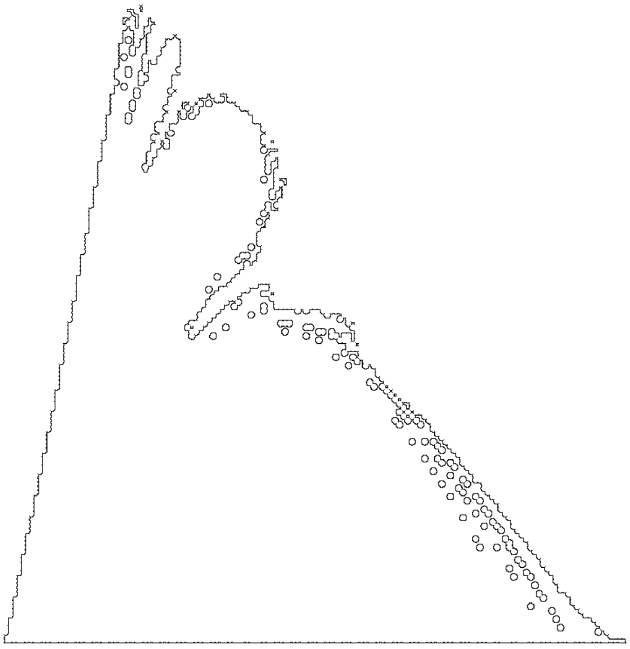


(m) $G = 0.160$

(n) $G = 0.165$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

(o) $G = 0.170$



(p) $G = 0.175$

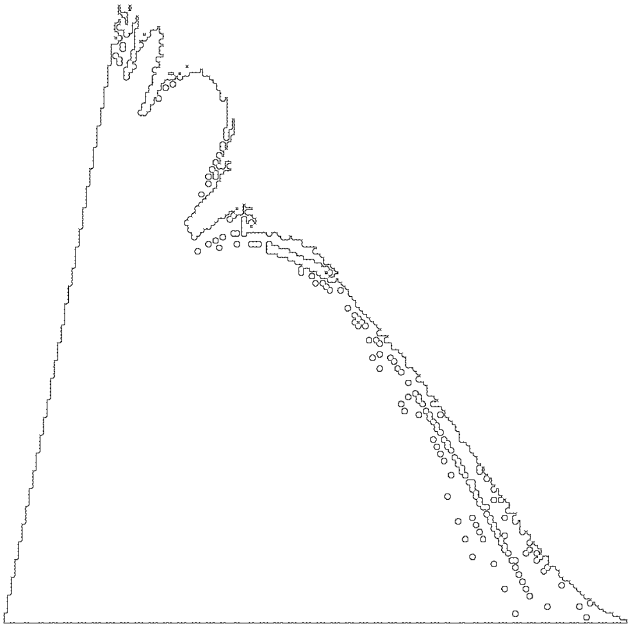
Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

(q) $G = 0.180$

(r) $G = 0.185$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

(s) $G = 0.190$



(t) $G = 0.195$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

(u) $G = 0.200$

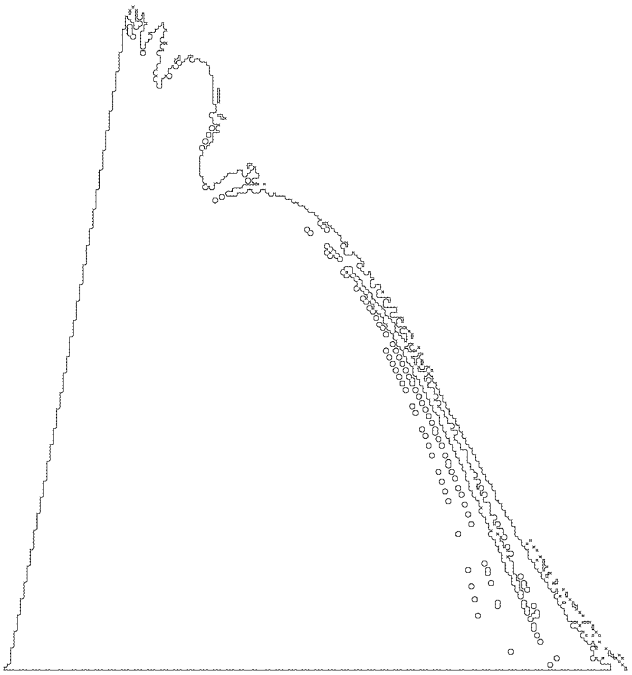
(v) $G = 0.205$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

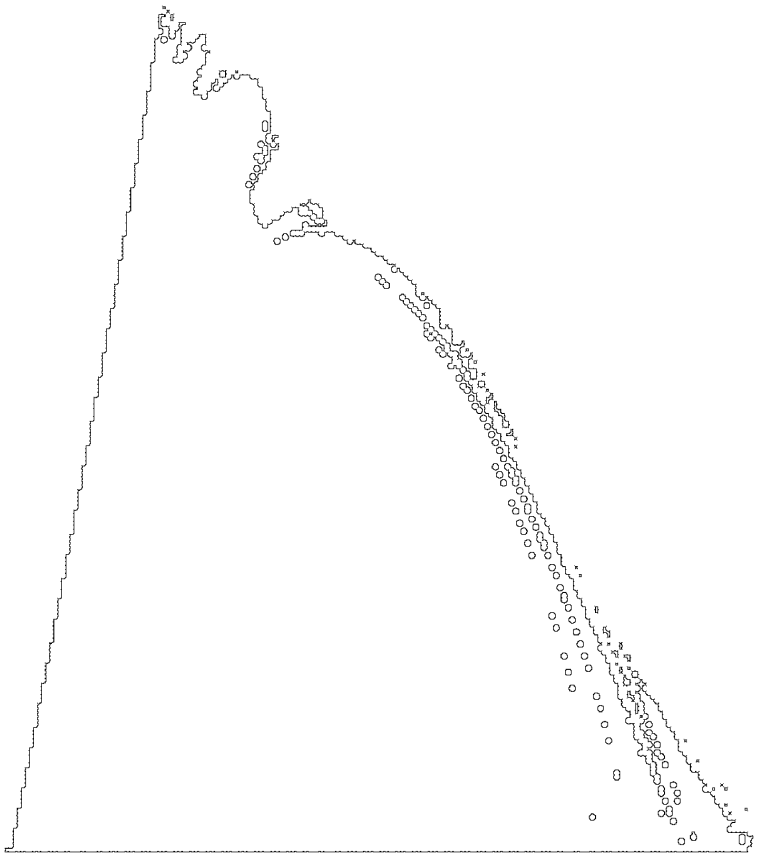
(w) $G = 0.210$

(x) $G = 0.215$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

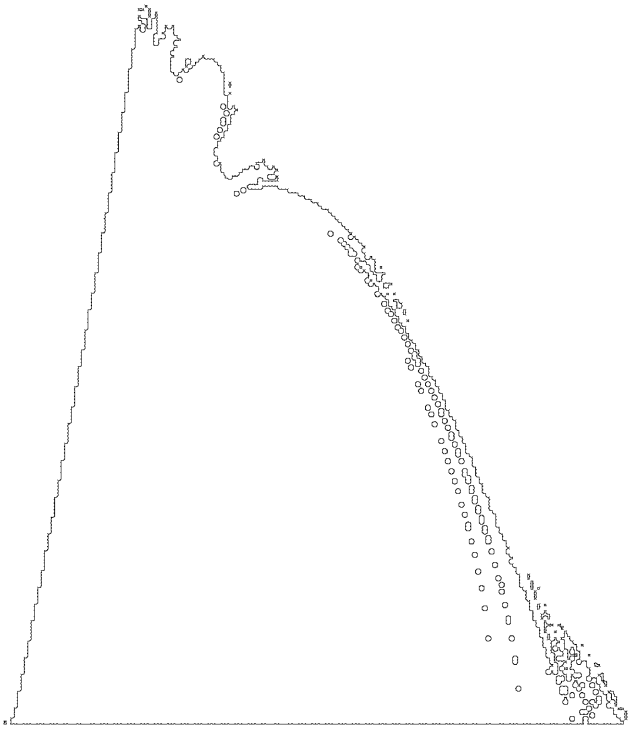


(y) $G = 0.220$



(z) $G = 0.225$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

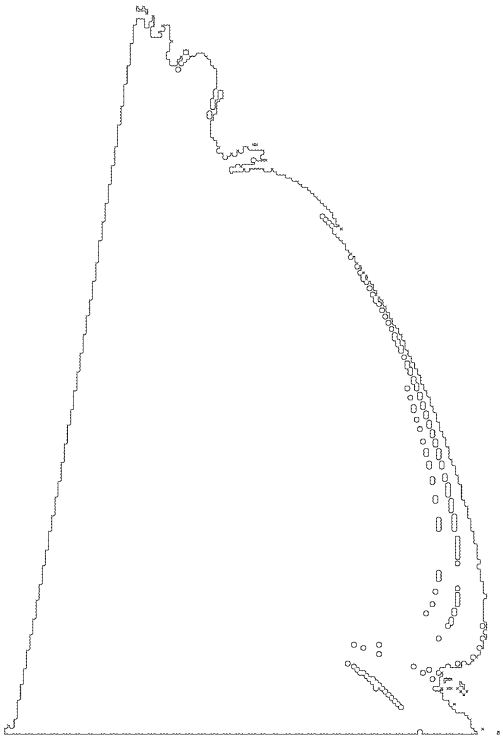


(aa) $G = 0.230$

(ab) $G = 0.235$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)

(ac) $G = 0.240$



(ad) $G = 0.245$

(ae) $G = 0.250$

Figure S11: Fractal boundary between inter-well and intra-well motion (two-colored damping ratio-forcing frequency parameter space)