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Harmonically induced shape morphing of bistable buckled beam with static bias

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ABSTRACT

We investigate the effect of a constant static bias force on the dynamically induced shape morphing of a prebuckled bistable beam, focusing on the beam's ability to change its vibration to be near different stable states under harmonic excitation. Our study explores four categories of oscillatory motions: switching, reverting, vacillating, and intra-well in the parameter space. We aim to achieve transitions between stable states of the pre-buckled bistable beam with minimal excitation amplitude. Our findings demonstrate the synergistic effects between dynamic excitation and static bias force, showing a broadening of the non-fractal region for switching behavior (i.e., switching from the first stable state to the second stable state) in the parameter space. This study advances the understanding of the dynamics of key structural components for multi-stable mechanical metamaterials, offering new possibilities for novel designs in adaptive applications.

Bistable buckled beams find application across mechanical metamaterials [1–5], energy harvesters [6–10], programmable mechanical devices [11,12], energy absorbers [13,14], and MEMS devices [15-17]. Bistable systems, known for their two stable equilibrium states, offer a promising avenue for applications that require morphing [18,19] and reconfiguration [20-24]. However, quasi-static shape morphing of bistable buckled beams is known to be energy-intensive and timeconsuming [25-30]. Vibration-induced shape morphing offers an alternative to quasi-static methods, requiring lower actuation amplitude and leveraging the inherent nonlinear dynamics of bistable systems [31-33]. By utilizing dynamic excitation, vibration-induced shape morphing effectively facilitates switching, reverting, and vacillating behavior, resulting in rapid and energy-efficient shape morphing [32]. Nonetheless, the parameter space of a symmetric bistable system reveals that switching and reverting behaviors coexist in an intertwined chaotic region [32,34–39]. This coexistence creates challenges in selecting the appropriate combination of forcing amplitude and frequency to switch between stable states. By exploiting asymmetric bistability [40-42], the switching behavior region within the parameter space of a bistable buckled beam can be broadened, thereby enhancing the feasibility of using dynamic excitation for shape morphing. Recent research demonstrates that we can tune the quasi-static response of magnetized buckled beams under applied magnetic fields-specifically, their quasi-static

force–displacement curves and energy landscapes exhibit asymmetric bistable behavior [43–46]. The translation of the force–displacement curve up or down along the force axis is linearly proportional to the applied magnetic field [43,44].

In this letter, we apply a static bias force in combination with dynamic excitation, exploiting asymmetric bistability to morph the bistable buckled beam between stable states. Our investigation focuses on enhancing the morphing of bistable beams through the synergistic combination of static bias forces and low-amplitude dynamic excitation to switch between the states of the bistable buckled beam. By manipulating the energy landscape using a static bias force, we aim to achieve faster and more reliable control over the bistable buckled beam's dynamic transitions, thereby broadening the switching behavior region in the parameter space of a bistable buckled beam and increasing the predictability of shape morphing. Our approach advances the theoretical understanding of bistable beam dynamics and opens new avenues for efficient shape morphing of bistable systems.

Initially, we consider a straight beam. We apply a compressive axial load that exceeds the critical buckling load, \hat{P}_{cr} (see Eq. (1)), to the left end as depicted in Fig. 1(a),

$$\hat{P}_{\rm cr} = \frac{4\pi^2 EI}{L^2},\tag{1}$$

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Fig. 1. Shape morphing of a bistable buckled beam exploiting symmetric bistability and dynamic excitation: (a) Schematic diagram of the beam's initial straight configuration. (b) Initially, the straight beam is compressed past the critical buckling load, \hat{P}_{err} to achieve a static deflection denoted as \hat{W}_0 with the fixed end. It can subsequently be dynamically excited to switch between its two stable states. (c) Analytical and finite element analysis (FEA) results showing switching behavior with dimensionless parameters $(G, \Omega) = (0.100, 1.17595)$, reverting behavior with parameters $(G, \Omega) = (0.125, 1.209412)$, aperiodic vacillating behavior with parameters $(G, \Omega) = (0.175, 1.13291)$, and intra-well behavior with parameters $(G, \Omega) = 0.07$. The results demonstrate good agreement between the analytical predictions and the FEA.

where *L*, *E*, and $I = \frac{bI^3}{12}$ represent the length, Young's modulus, and moment of inertia of the buckled beam, respectively. The compressive axial load buckles the beam to a static deflection position, defined by Eq. (2) [47],

$$\hat{W}_o(\hat{\mathbf{x}}) = \frac{h}{2} \left\{ 1 - \cos\left(2\pi \frac{\hat{\mathbf{x}}}{L}\right) \right\}.$$
(2)

We then fix both ends of the beam, resulting in a curved beam, as shown in Fig. 1(b). The initial static deflection of the beam is defined by Eq. (2) and the vertical rise of the midpoint is given by $\hat{W}_o(x = \frac{L}{2}) = h$, where *h* represents the amplitude of the first buckled mode shape of the clamped-clamped beam.

The beam has two stable states (Fig. 1(b)). We can induce a switch between its stable states by applying harmonic excitation at the midpoint, given by $\hat{W}_o(x = \frac{L}{2}) = h$, as illustrated in Fig. 1(b). To model the bistable buckled beam analytically, we begin with the nonlinear Euler-Bernoulli beam equation, which is represented by Eq. (3) [48],

$$EI\frac{\partial^{4}W}{\partial\hat{x}^{4}} + \hat{P}_{cr}\frac{\partial^{2}W}{\partial\hat{x}^{2}} + \rho A\frac{\partial^{2}W}{\partial\hat{t}^{2}} + \hat{C}^{d}\frac{\partial W}{\partial\hat{t}} - \frac{EA}{2L}\left\{\int_{0}^{L}\left[\left(\frac{\partial\hat{W}}{\partial\hat{x}}\right)^{2} + 2\frac{\partial\hat{W}}{\partial\hat{x}}\frac{\partial\hat{W}_{o}}{\partial\hat{x}}\right]d\hat{x}\right\} (3) \times \left(\frac{\partial^{2}\hat{W}}{\partial\hat{x}^{2}} + \frac{\partial^{2}\hat{W}_{o}}{\partial\hat{x}^{2}}\right) = \hat{F}\cos\left(\hat{\Omega}\hat{t}\right).$$

By approximating the first buckling mode using Eq. (2) and applying Galerkin approximation, we discretize Eq. (3) into Eq. (4), which represents the symmetric bistable Duffing equation [49] (see supplemental materials [49] for detailed derivation).

$$\ddot{u} + \gamma \dot{u} - u + u^3 = G \cos{(\Omega \tau)},\tag{4}$$

where G, Ω , γ are the non-dimensionalized forcing amplitude, excitation frequency, and damping ratio, respectively (see supplemental materials [49] for derivations). Eq. (4) depicts the dimensionless symmetric bistable Duffing equation, characterized by double-well potential with two stable equilibrium points at $u_{-1} = -1$ and $u_{+1} = +1$. These points are separated by an unstable equilibrium, or "hilltop", at $u_0 = 0$ [32]. Fig. 1(c) illustrates four distinct behaviors: switching, reverting, vacillating, and intra-well. Our previous study established numerical criteria to distinguish among these behaviors by conducting time-domain simulations on Eq. (4) [32]. We fix the initial conditions in all simulations at $(u, \dot{u}) = (-1, 0)$. Here, we verify four distinct behaviors by conducting finite element analysis using Abaqus/Standard. We normalize all the dimensions using the initial vertical rise of the beam $\hat{W}_o(x = \frac{L}{2}) = h = 5.22$ mm. Initially, we model a straight beam with normalized dimensions: length L/h = 11.5, thickness t/h = 0.2, and width b/h = 1.92. This beam undergoes buckling when subjected to a load exceeding the critical buckling load, which is $\frac{\hat{P}_{cr}L^2}{r} = 39.478$ (see supplemental materials [49] for detailed derivation). Subsequently, we perform a modal analysis on the buckled beam in Abagus/Standard to identify its first buckling mode. We use B21 elements and a hyperelastic material, Dragon Skin 30 [50], which is nearly incompressible (Poisson's ratio $v \approx 0.495$), with an initial Young's modulus of $E = 0.74 \pm 0.07$ MPa [51], to model the bistable buckled beam.

Then, we perform dynamic implicit analysis to confirm the four types of behavior predicted analytically: switching behavior with dimensionless parameters $(G, \Omega) = (0.100, 1.17595)$, reverting behavior with parameters (G, Ω) = (0.125, 1.209412), aperiodic vacillating behavior with parameters $(G, \Omega) = (0.175, 1.13291)$, and intra-well behavior with parameters (G, Ω) = (0.150, 1.30506), all under a constant damping ratio of $\gamma = 0.07$. Fig. 1(c) demonstrates good agreement between analytical and FEA results for these behaviors. The static component of Eq. (4), $F_{\text{static}} = -u + u^3$, characterizes an energy landscape with equal potential wells, requiring a substantial static actuation force of 0.38 units to transition the bistable buckled beam from one stable state to another [32]. Fig. 1(c) demonstrates that dynamic excitation of the bistable system can significantly reduce the required actuation force compared with quasi-static actuation [32]. In a previous study, we presented the forcing amplitude-frequency parameter space for a symmetric bistable Duffing system at a constant damping ratio of $\gamma = 0.07$ [32]. The findings revealed that although low forcing amplitudes can produce four distinct behaviors, the simultaneous existence of switching, reverting, and vacillating behaviors near each other in this parameter space leads to an intertwined chaotic region. The intertwined chaotic region in the parameter space complicates the selection of an appropriate switching frequency and forcing amplitude for transitioning the bistable buckled beam between its stable states.

Recent studies have demonstrated that applying a magnetic field can programmatically alter the energy landscape of a magnetized bistable buckled beam, allowing for the tuning of stability characteristics and modification of the energy landscape through static analysis under varying magnetic fields [43–46]. In particular, the forcedisplacement curve experiences a linear translation toward one of the stable states, dictated by the direction of the applied magnetic field [43,44].

Building on this concept, we apply a static bias force P(B) to linearly shift the force–displacement curve along the force axis, altering the system's energy landscape. By adjusting P(B), we can manipulate the stability characteristics of the beam, achieving a controlled shift similar to the effects observed with magnetic fields. Incorporating P(B) as a static bias force changes Eq. (4) to Eq. (5), altering the system's



Fig. 2. Tuning stability characteristics of force–displacement and energy landscapes under varying static bias forces P(B): (a) Evolution of the force–displacement curve, showing linear shifts along the force axis in response to varying levels of static bias force, P(B). (b) The transition from symmetric bistable to asymmetric bistable energy landscapes with increasing static bias force. (c) A linear relationship between the absolute value of $F_{\min,static}$ and the static bias force P(B). (d) A demonstration of combined static bias force and dynamic excitation to the bistable buckled beam.

response. Eq. (5) is the dimensionless asymmetric bistable Duffing equation, characterized by an asymmetric double-well potential with two stable equilibrium points. The impact of the static bias force P(B)on the symmetric bistable system is understood through the static component of Eq. (5), $F_{\text{static}} = -u + u^3 - P(B)$. This influence is visually demonstrated in Figs. 2(a) and (b), which show how the force-displacement and energy-displacement curves change with the application of P(B) (see supplemental materials [49] for more results). Specifically, the potential well at the stable equilibrium point $u_{-1} = -1$ increases, while the potential well at $u_{+1} = +1$ decreases. Fig. 2(c) presents the absolute value of $F_{\text{min,static}}$, the minimum force in the second stable state, for each force-displacement curve across varying values of P(B), demonstrating a linear relationship between absolute value of $F_{\min,static}$ and the static bias force P(B). Now, we can apply a combined static bias force with dynamic excitation to switch the bistable buckled beam from one stable state to another, as demonstrated in Fig. 2(d).

$$\ddot{u} + \gamma \dot{u} - u + u^3 = G \cos\left(\Omega \tau\right) + P(B), \tag{5}$$

We next conduct a time-domain simulation using the fourth-order Runge–Kutta scheme to solve Eq. (5) for P(B) = 0.100 with parameters of $(G, \Omega) = (0.125, 1.209412)$ of the reverting behavior of Fig. 1(c) and a damping ratio of $\gamma = 0.07$. Fig. 3(a) presents the simulation results, demonstrating that under the influence of a static bias force, the reverting behavior depicted in Fig. 1(c) transitions to the switching behavior. Similarly, for the vacillating behavior with parameters $(G, \Omega) = (0.175, 1.13291)$ and a damping ratio of $\gamma = 0.07$, the static bias force of P(B) = 0.100 changes the vacillating behavior of Fig. 1(c) into the switching behavior, as shown in Fig. 3(b).

Figs. 3(c)–(f) display the forcing amplitude–frequency parameter space within the ranges $0.80 \le \Omega \le 1.8$ and $0.03 \le G \le 0.30$, with a damping ratio of $\gamma = 0.07$ and static bias forces P(B) = 0.04, 0.100, 0.200, 0.300 respectively. For each combination of (G, Ω) , we



Fig. 3. Morphing of the bistable buckled beam's behavior due to combined dynamic excitation and static bias force: (a) The reverting behavior depicted in Fig. 1(c) transitions to switching behavior when subjected to both dynamic excitation and a static bias force. (b) Similarly, the vacillating behavior shown in Fig. 1(c) shifts to switching behavior under the same combined forces. Panels (c), (d), (e), and (f) display the forcing amplitude–frequency parameter space for the bistable buckled beam with a damping ratio of $\gamma = 0.07$ and static bias forces of P(B) = 0.04, 0.100, 0.200, 0.300 respectively. The light gray \bigcirc and \triangle in Fig. 3(d) indicate the points where the reverting behavior with parameters (*G*, Ω) = (0.125, 1.209412) and the aperiodic vacillating behavior with pehavior.

conduct time-domain simulations of Eq. (5) across these parameter ranges, incorporating the aforementioned damping ratios and static bias forces. Figs. 3(c)–(f) present the results for P(B) = 0.04, 0.100, 0.200, 0.300, plotted on a 256 \times 256 grid (for additional results with other P(B) values, see supplemental materials [49]). We classify each simulation's numerical steady state into four behaviors: switching, reverting, vacillating, or intra-well, represented by red, blue, green, and yellow data points in Figs. 3(c)-(f), respectively. This categorization follows the methodology established in our previous study [32]. One notable observation from Figs. 3(c)-(f) is that as the static bias force increases, the switching behavior becomes more prominent across all parameter spaces compared to other behaviors (see supplemental materials [49] for more results). Furthermore, as P(B) increases, the minimum dynamic forcing amplitude required to switch between stable states in both directions decreases. This implies that a higher static bias not only makes the potential well asymmetric but also reduces the dynamic forcing amplitude requirement, denoted as G_{\min} , which is the minimum for the possibility of switching.

Next, we quantify the switching behavior area within the parameter spaces of Figs. 3(c)–(f) where the parameter set (G, Ω) always results in switching behavior. Fig. 4(a) illustrates a parameter space with static bias force P(B) = 0.160 where a rectangle indicates the area where no behaviors other than switching are present. We quantify this area from a 256 × 256 grid, where each set of (G, Ω) results in switching



Fig. 4. Distinct switching areas: (a) shows a parameter space for P(B) = 0.160 and $\gamma = 0.07$, outlined by a rectangle, where every combination of *G* and Ω results in switching behavior. (b) illustrates the expansion of the switching behavior area as the static bias force *P*(B) increases across different parameter spaces.

behaviors. We numerically detect the largest possible rectangle in the parameter space, excluding other behaviors. After measuring the width and height of the rectangle across the frequency and forcing amplitude ranges, we calculate the distinct dimensionless switching area, $\Delta\Omega \times \Delta G$, where no other behaviors are present. Fig. 4(b) shows that as we increase the static bias force, the dimensionless rectangular area in the parameter spaces increases (see supplemental materials [49] for additional results).

In conclusion, our investigation shows the dynamic morphing capabilities of bistable buckled beams under the influence of static bias forces and dynamic excitation. We demonstrate that applying a static bias force expands the parameter space conducive to switching behavior, thus facilitating more efficient transitions between stable states without the onset of chaos. This enhancement of the switching behavior region underscores the potential of static bias force as a tool for optimizing the morphing efficiency of bistable systems.

CRediT authorship contribution statement

Md Nahid Hasan: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. Sharat Paul: Writing – review & editing, Writing – original draft, Visualization, Validation, Investigation, Formal analysis, Data curation. Taylor E. Greenwood: Writing – review & editing, Supervision. Robert G. Parker: Writing – review & editing, Validation, Investigation. Yong Lin Kong: Writing – review & editing, Supervision, Software, Resources, Project administration, Funding acquisition. Pai Wang: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: All authors report financial support was provided by National Institutes of Health. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.eml.2025.102299.

In the supplemental materials, we have included a detailed derivation of the bistable Duffing equation from the nonlinear vibration equation of the buckled beam. Furthermore, we have added additional results of Figs. 2–4. We acknowledge the support from the National Institutes of Health (NIH): Project No. R01EB032959. Start-up funds from the Department of Mechanical Engineering at the Univ. of Utah also supported this work. The support and resources from the Center for High-Performance Computing at Univ. of Utah are gratefully acknowledged.

Data availability

Data will be made available on request.

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Supplementary Materials for Harmonically Induced Shape Morphing of Bistable Buckled Beam with Static Bias

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I. CONTINUOUS BUCKLED BEAM EQUATION TO DISCRETE BISTABLE DUFFING EQUATION

We consider a straight beam buckled into its first buckled mode and apply harmonic excitation to switch its state from the first stable state to the second stable state, as shown in Figs. 1(a) and 1(b) in the main manuscript. Furthermore, We consider the transverse deflection of the beam at a position x and time t as $\hat{W}(x,t)$. The beam has a length L, a uniform density ρ , a cross-sectional area A, and a flexural rigidity EI, where E is Young's modulus and $I = \frac{bt^3}{12}$ is the moment of inertia of the beam. Here, t and b are the thickness and the out-of-plane width of the beam, respectively (see Table I). Initially, the bistable beam is modeled using a continuous beam vibration equation, which is then discretized into a bistable Duffing equation. The governing differential equation for a buckled beam, originally flat and then compressed past its critical buckling load, results in a static displacement \hat{W}_o when the ends are fixed. This beam is subjected to a point load harmonic excitation (Figs. 1(a) and 1(b) in the main manuscript). Equation S1 shows the Euler-Bernoulli beam equation of a buckled beam.

$$EI\frac{\partial^{4}\hat{W}}{\partial\hat{x}^{4}} + \hat{P}_{cr}\frac{\partial^{2}\hat{W}}{\partial\hat{x}^{2}} + \rho A\frac{\partial^{2}\hat{W}}{\partial\hat{t}^{2}} + \hat{C}^{d}\frac{\partial\hat{W}}{\partial\hat{t}} - \left\{\frac{EA}{2L}\int_{0}^{L}\left[\left(\frac{\partial\hat{W}}{\partial\hat{x}}\right)^{2} + 2\frac{\partial\hat{W}}{\partial\hat{x}}\frac{\partial\hat{W}_{o}}{\partial\hat{x}}\right]d\hat{x}\right\}$$
(S1)
$$\left(\frac{\partial^{2}\hat{W}}{\partial\hat{x}^{2}} + \frac{\partial^{2}\hat{W}_{o}}{\partial\hat{x}^{2}}\right) = \hat{F}\cos\left(\hat{\Omega}\hat{t}\right).$$

Equation (S1) can be nondimensionalized using the nondimensionalized parameter from the Table II,

$$\frac{\partial^4 W}{\partial x^4} = \frac{\partial^4(\hat{W}h)}{\partial(\hat{x}L)^4} = \frac{h}{L^4} \frac{\partial^4 \hat{W}}{\partial \hat{x}^4}; \quad \frac{\partial^2 W}{\partial x^2} = \frac{\partial^2(\hat{W}h)}{\partial(\hat{x}L)^2} = \frac{h}{L^2} \frac{\partial^2 \hat{W}}{\partial \hat{x}^2}; \quad \frac{\partial W}{\partial x} = \frac{\partial(\hat{W}h)}{\partial(\hat{x}L)} = \frac{h}{L} \frac{\partial \hat{W}}{\partial \hat{x}}; \\ \left(\frac{\partial W_o}{\partial x}\right)^2 = \frac{h^2}{L^2} \left(\frac{\partial \hat{W}_o}{\partial \hat{x}}\right)^2; \quad \left(\frac{\partial W_o}{\partial x}\right) = \frac{h}{L} \left(\frac{\partial \hat{W}_o}{\partial \hat{x}}\right); \quad \frac{\partial^2 W_o}{\partial x^2} = \frac{\partial^2(\hat{W}_oh)}{\partial(\hat{x}L)^2} = \frac{h}{L^2} \frac{\partial^2 \hat{W}_o}{\partial \hat{x}^2}; \\ \frac{\partial W}{\partial t} = \frac{h}{T} \frac{\partial \hat{W}}{\partial \hat{t}}; \quad \frac{\partial^2 W}{\partial t^2} = \frac{\partial^2(\hat{W}h)}{\partial(\hat{t}T)^2} = \frac{h}{T^2} \frac{\partial^2 \hat{W}}{\partial \hat{t}^2}; \quad dx = d(\hat{x}L) = Ld\hat{x}, \end{cases}$$
(S2)

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where $\hat{W}_o(x = \frac{L}{2}) = h$ is the apex height or amplitude of the bistable buckled beam and replace the above substitution in Eq. (S1) we get,

$$\frac{EIh}{L^4} \frac{\partial^4 W}{\partial x^4} + \frac{\hat{P}_{cr}h}{L^2} \frac{\partial^2 W}{\partial x^2} + \frac{\rho Ah}{T^2} \frac{\partial^2 W}{\partial t^2} + \frac{\hat{C}^d h}{T} \frac{\partial W}{\partial t} - \left\{ \frac{EAh^3}{2L^4} \int_0^L \left[\left(\frac{\partial W}{\partial x} \right)^2 + 2 \frac{\partial W}{\partial x} \frac{\partial W_o}{\partial x} \right] dx \right\} \\ \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W_o}{\partial x^2} \right) \\ = \hat{F} \cos(\hat{\Omega}Tt).$$
(S3)

Now divide both sides of the Eq. (S3) by, $\frac{EIh}{L^4}$

$$\frac{\partial^4 W}{\partial x^4} + \frac{\hat{P}_{cr}L^2}{EI} \frac{\partial^2 W}{\partial x^2} + \frac{\rho A L^4}{EIT^2} \frac{\partial^2 W}{\partial t^2} + \frac{\hat{C}^d L^4}{EIT} \frac{\partial W}{\partial t} - \left\{ \frac{A h^2}{2I} \int_0^L \left[\left(\frac{\partial W}{\partial x} \right)^2 + 2 \frac{\partial W}{\partial x} \frac{\partial W_o}{\partial x} \right] dx \right\} \\ \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W_o}{\partial x^2} \right) \\ = \frac{\hat{F} L^4}{EIh} \cos(\hat{\Omega} T t),$$
(S4)

Next, T is defined by setting the coefficient of the inertia term equal to unity,

$$\frac{\rho A L^4}{E I T^2} = 1 \implies T = \sqrt{\frac{\rho A L^4}{E I}},\tag{S5}$$

T is called the time constant. Using the nondimensional substitutions from Table II and Eq. (S5), Eq. (S4) becomes Eq. (S6).

$$\frac{\partial^4 W}{\partial x^4} + P_{cr} \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial t^2} + C^d \frac{\partial W}{\partial t} - \left\{ 6Q^2 \int_0^L \left[\left(\frac{\partial W}{\partial x} \right)^2 + 2 \frac{\partial W}{\partial x} \frac{\partial W_o}{\partial x} \right] dx \right\} \\ \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W_o}{\partial x^2} \right) \qquad (S6) \\ = F \cos(\hat{\Omega} T t).$$

To solve Eq. (S6), we apply the separation-of-variables method, considering a solution as the product of a spatial function $\phi_i(x)$ and a time-dependent function $q_i(t)$. Using Galerkin's method, Eq. (S6) is transformed into a set of coupled ordinary differential equations (ODEs). With *n* representing the degrees of freedom (DOF), the separation of variables on Eq. (S6) leads to:

$$W(x,t) = \sum_{i=1}^{n} q_i(t)\phi_i(x).$$
 (S7)

Parameter	Symbol	Value
Beam length (mm)	L	60
Beam width (mm)	b	10
Beam thickness (mm)	t	1
Buckled height (mm)	h	5.22
Modulus of elasticity (MPa)	E	0.74 ± 0.07
Density $\left(\frac{kg}{m^3}\right)$	ρ	1082

TABLE I. Beam Geometry and Material Properties (Dragon Skin 30)

Mode shape of the bistable buckled beam is given by,

Odd mode,

$$\phi_i(x) = \frac{1}{2} \left[1 - \cos(N_i x) \right],$$
(S8)

$$N_i = (i+1)\pi\tag{S9}$$

Here, i = 1, 3, 5,

and even modes,

$$\phi_i(x) = \frac{1}{2} \left[1 - 2x - \cos(N_i x) + \frac{2\sin(N_i x)}{N_i} \right],$$
(S10)

$$N_i = 2.86\pi, 4.92\pi.... \tag{S11}$$

Here, $i = 2, 4, 6, \dots$ Now plugging Eq. (S7) into the Eq. (S6) yields a coupled set of n ODE's for q_i .

$$\sum_{i=1}^{n} \phi_{i} \frac{\partial^{2} q_{i}}{\partial t^{2}} + C^{d} \sum_{i=1}^{n} \phi_{i} \frac{\partial q_{i}}{\partial t} + P_{cr} \sum_{i=1}^{n} \frac{\partial^{2} \phi_{i}}{\partial x^{2}} q_{i} + \sum_{i=1}^{n} \frac{\partial^{4} \phi_{i}}{\partial x^{4}} q_{i}$$
$$- \left\{ 6Q^{2} \int_{0}^{L} \left[\left(\frac{\partial \phi_{i}}{\partial x} \right)^{2} q_{i}^{2} + 2 \left(\frac{\partial \phi_{i}}{\partial x} \right) \left(\frac{d\phi_{o}}{dx} \right) q_{i} \right] dx \right\} \left(\frac{\partial^{2} \phi_{i}}{\partial x^{2}} q_{i} + \frac{d^{2} \phi_{o}}{dx^{2}} \right) = F \cos\left(\hat{\Omega}T\hat{t}\right).$$
(S12)

As the buckling mode shapes are orthogonal, the linear terms in the Eq. (S12) can be decoupled by multiplying through ϕ_j and integrating over the length of the beam. This provides

Parameter	Substitutions
x- direction position	$x = \frac{\hat{x}}{L}$
z- direction position	$w = \frac{\hat{w}}{h}$
Time	$t = \frac{\hat{t}}{T}$
Damping Coefficient	$C^d = \frac{\hat{C}^d L^2}{\sqrt{\rho A E I}}$
Axial Load	$P_{cr} = \hat{P}_{cr} \frac{L^2}{EI}$
Force	$F = \frac{\hat{F}L^3}{EIh}$
Geometric parameter	$Q = \left(\frac{h}{t}\right)$
Time constant (s)	$T = \sqrt{\frac{\rho A L^4}{EI}}$

TABLE II. Nondimentional substitution

a set of ordinary differential equations,

$$M_{i}\ddot{q}_{i} + C^{d}M_{i}\dot{q}_{i} + P_{cr}E_{i}q_{i} + N_{i}q_{i} - 6Q^{2}\left[\left(D_{i}q_{i}^{2} + 2G_{i}q_{i}\right)\right]\left(E_{i}q_{i} + H_{i}\right) = F_{i}F\cos\left(\hat{\Omega}Tt\right),$$
(S13)

where

$$M_{i} = \int_{0}^{1} \phi_{j} \phi_{i} dx; \quad N_{i} = \int_{0}^{1} \phi_{j} \frac{d^{4} \phi_{i}}{dx^{4}} dx; \quad D_{i} = \int_{0}^{1} \left(\frac{d\phi_{i}}{dx}\right)^{2} dx;$$

$$E_{i} = \int_{0}^{1} \phi_{j} \frac{d^{2} \phi_{i}}{dx^{2}} dx; \quad F_{i} = \int_{0}^{1} \phi_{j} dx; \quad G_{i} = \int_{0}^{1} \left(\frac{d\phi_{i}}{dx}\right) \left(\frac{d\phi_{o}}{dx}\right) dx; \quad H_{i} = \int_{0}^{1} \phi_{j} \frac{d^{2} \phi_{o}}{dx^{2}} dx.$$
(S14)

Mode shapes ϕ_i and ϕ_j are orthogonal to each other, a key concept in structural dynamics. The orthonormality condition for these mode shapes is defined as follows:

$$\int_0^1 \phi_i \phi_j \, dx = \delta_{ij},\tag{S15}$$

The Kronecker delta, δ_{ij} , signifies the orthonormality and is defined as:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$
(S16)

This orthonormality condition ensures that the integral of the product of two different mode shapes, ϕ_i and ϕ_j , over their domain is zero when $i \neq j$, and is equal to 1 when i = j.

If we consider the first buckled mode approximation of Eq. (S8),

$$\phi_1(x) = \frac{1}{2} \left[1 - \cos(2\pi x) \right], \tag{S17}$$

Therefore, first Buckling mode parameters,

$$M_{1} = \int_{0}^{1} \phi_{1}\phi_{1} = 0.3750; \quad N_{1} = \int_{0}^{1} \phi_{1}\frac{d^{4}\phi_{1}}{dx^{4}} = 194.8182;$$
$$D_{1} = \int_{0}^{1} \left(\frac{d\phi_{1}}{dx}\right)^{2} = 4.9348; \quad E_{1} = \int_{0}^{1} \phi_{1}\frac{d^{2}\phi_{1}}{dx^{2}} = -4.9348; \quad F_{1} = \int_{0}^{1} \phi_{1} = 0.500; \quad (S18)$$
$$G_{1} = \int_{0}^{1} \left(\frac{d\phi_{1}}{dx}\right) \left(\frac{d\phi_{o}}{dx}\right) = 4.9348; \quad H_{1} = \int_{0}^{1} \phi_{1}\frac{d^{2}\phi_{o}}{dx^{2}} = -4.9348,$$

Therefore, Eq. (S13) becomes,

$$M_1 \ddot{q}_1 + C^d M_1 \dot{q}_1 + P_{cr} E_1 q_1 + N_1 q_1 - 6Q^2 \left[(D_1 q_1^2 + 2G_1 q_1) \right] (E_1 q_1 + H_1)$$

$$= F_1 F \cos(\hat{\Omega} T t),$$
(S19)

$$M_{1}\ddot{q}_{1} + C^{d}M_{1}\dot{q}_{1} + P_{cr}E_{1}q_{1} + N_{1}q_{1} - 6Q^{2}\left[D_{1}E_{1}q_{1}^{3} + D_{1}H_{1}q_{1}^{2} + 2E_{1}G_{1}q_{1}^{2} + 2G_{1}H_{1}q_{1}\right]$$

= $F_{1}F\cos\left(\hat{\Omega}Tt\right),$
(S20)

As
$$D_1 = G_1$$
, and $E_1 = H_1$, Eq. (S20) simplifies,
 $M_1\ddot{q}_1 + C^d M_1 \dot{q}_1 + P_{cr} E_1 q_1 + N_1 q_1 - 6Q^2 \left[D_1 E_1 q_1^3 + D_1 E_1 q_1^2 + 2D_1 E_1 q_1^2 + 2D_1 E_1 q_1 \right]$

$$= F_1 F \cos(\hat{\Omega} T t),$$
(S21)

$$M_1 \ddot{q}_1 + C^d M_1 \dot{q}_1 + P_{cr} E_1 q_1 + N_1 q_1 - 6Q^2 \left[D_1 E_1 q_1^3 + 3D_1 E_1 q_1^2 + 2D_1 E_1 q_1 \right]$$

= $F_1 F \cos(\hat{\Omega} T t),$ (S22)

As $(N_1 + P_{cr}E_1) = (194.8182 - 194.8181) \approx 0$,

$$M_1 \ddot{q}_1 + C^d M_1 \dot{q}_1 - 6Q^2 \left[D_1 E_1 q_1^3 + 3D_1 E_1 q_1^2 + 2D_1 E_1 q_1 \right] = F_1 F \cos\left(\hat{\Omega} T t\right),$$
(S23)

and let, $K = -6Q^2D_1E_1$ with $D_1 = 4.9348$ and $E_1 = -4.9348$ and Q = h/t, so the final form of the Eq. (S20),

$$M_1 \ddot{q}_1 + C^d M_1 \dot{q}_1 + \left[K q_1^3 + 3K q_1^2 + 2K q_1 \right] = F_1 F \cos\left(\hat{\Omega} T t\right).$$
(S24)

We shift the equilibrium to zero for the new variable u using the transformation $q_1 = u - 1$. This simplifies the equations, helps linearize the system, and makes analysis and computations easier. It aligns with standard approximation methods, enhancing both analytical and numerical analysis. For the given equation, we apply this transformation as follows:

$$u = q_1 + 1 \quad \Longrightarrow \quad q_1 = u - 1, \tag{S25}$$

Using the above equation Eq. (S24) becomes,

$$M_1 \ddot{u} + C^d M_1 \dot{u} - K u + K u^3 = F_1 F \cos(\hat{\Omega} T t),$$
 (S26)

$$\ddot{u} + C^{d}\dot{u} - \frac{K}{M_{1}}u_{1} + \frac{K}{M_{1}}u^{3} = F_{1}\left(\frac{F}{M_{1}}\right)\cos\left(\hat{\Omega}Tt\right),$$
(S27)

$$\ddot{u} + C^d \dot{u} - \omega_{non}^2 u + \omega_{non}^2 u^3 = F_1 \left(\frac{F}{M_1}\right) \cos\left(\hat{\Omega}Tt\right).$$
(S28)

 ω_{non} is the first natural frequency of the first mode at the linear limit,

$$\omega_{non}^2 = \left(\frac{K}{M_1}\right) = 4\pi^4 Q^2. \tag{S29}$$

We scale the time to transform Eq. (S28) into the standard form of a bistable Duffing equation by introducing the scaled time variable $\tau = \omega_{\text{non}} t$, where ω_{non} is the first natural frequency of the system at the linear limit. We can write Eq. (S28) like this,

$$\frac{d^2u}{dt^2} + C^d \frac{du}{dt} - \omega_{non}^2 u + \omega_{non}^2 u^3 = F_1\left(\frac{F}{M_1}\right)\cos\left(\hat{\Omega}Tt\right).$$
(S30)

Let
$$g = F_1\left(\frac{F}{M_1}\right)$$
,
$$\frac{d^2u}{dt^2} + C^d \frac{du}{dt} - \omega_{non}^2 u + \omega_{non}^2 u^3 = g\cos\left(\hat{\Omega}Tt\right),$$
(S31)

As,
$$\tau = \omega_{non} t$$
,

$$\omega_{non}^2 \frac{d^2 u}{d\tau^2} + \omega_{non} C^d \frac{du}{d\tau} - \omega_{non}^2 u + \omega_{non}^2 u^3 = g \cos\left(\frac{\hat{\Omega}T}{\omega_{non}}\tau\right),$$
(S32)

dividing both sides of the above equation by ω_{non}^2

$$\frac{d^2u}{d\tau^2} + \frac{C^d}{\omega_{non}}\frac{du}{d\tau} - u + u^3 = \frac{g}{\omega_{non}^2}\cos\left(\frac{\hat{\Omega}T}{\omega_{non}}\tau\right).$$
(S33)

We assume the nondimensional excitation frequency as $\omega = \frac{\hat{\Omega}T}{\omega_{\text{non}}}$. Let $G = \frac{g}{\omega_{\text{non}}^2}$ and $\gamma = \frac{C^d}{\omega_{\text{non}}}$. With these substitutions, Eq. (S33) becomes:

$$\ddot{u} + \gamma \dot{u} - u + u^3 = G \cos\left(\omega\tau\right),\tag{S34}$$

where G is the point modal force amplitude (projected force with respect to the first mode shape) considering time scaling,

$$G = F_1 \left(\frac{F}{4\pi^4 Q^2 \cdot M_1} \right),$$

with $F = \hat{F} \cdot \frac{L^3}{EIh}, \quad F_1 = \int_0^1 \phi_1 = 0.500, \quad Q = h/t,$ (S35)
and $M_1 = \int_0^1 \phi_1 \phi_1 = 0.3750.$

The nondimensional excitation frequency ω (considering time scaling and the time constant T) is given by:

$$\omega = \hat{\Omega} \cdot \sqrt{4\pi^4 Q^2} \cdot \sqrt{\frac{EI}{\rho A L^4}}.$$
(S36)

II. IMPACT OF STATIC BIAS FORCE P(B) ON THE SYMMETRIC BISTABLE SYSTEM



FIG. S1. Tunability of the bistable behavior: (a) Evolution of the force-displacement curve, demonstrating linear shifts along the force axis. (b) Linear translation of the energy landscape under various applied static bias forces. The black arrows indicate the direction of translation.

III. FORCING AMPLITUDE-FREQUENCY PARAMETER SPACE FOR A BUCKLED BEAM UNDER COMBINED STATIC BIAS FORCE AND DYNAMIC EXCITATION



FIG. S2. Forcing amplitude-frequency parameter space for a bistable buckled beam with a damping ratio of $\gamma = 0.07$ across static bias forces from P(B) = 0.00 to 0.100 in increments of 0.02.



FIG. S3. Forcing amplitude-frequency parameter space for a bistable buckled beam with a damping ratio of $\gamma = 0.07$ across static bias forces from P(B) = 0.120 to 0.200 in increments of 0.02.



FIG. S4. Forcing amplitude-frequency parameter space for a bistable buckled beam with a damping ratio of $\gamma = 0.07$ across static bias forces from P(B) = 0.240 to 0.300 in increments of 0.02.





FIG. S5. Forcing amplitude-frequency parameter space for a bistable buckled beam with a damping ratio of $\gamma = 0.07$ across static bias forces from P(B) = 0.02 to 0.120 in increments of 0.02. A distinct switching area is marked by a rectangle indicating where every combination of G and Ω results in switching behavior.



FIG. S6. Forcing amplitude-frequency parameter space for a bistable buckled beam with a damping ratio of $\gamma = 0.07$ across static bias forces from P(B) = 0.140 to 0.240 in increments of 0.02. A distinct switching area is marked by a rectangle indicating where every combination of G and Ω results in switching behavior.



FIG. S7. Forcing amplitude-frequency parameter space for a bistable buckled beam with a damping ratio of $\gamma = 0.07$ across static bias forces from P(B) = 0.240 to 0.300 in increments of 0.02. A distinct switching area is marked by a rectangle indicating where every combination of G and Ω results in switching behavior.

V. RELATIONSHIP BETWEEN STATIC BIAS FORCE AND THE MINIMUM FORCING AMPLITUDE G_{\min} IN THE PARAMETER SPACE, AND ITS RELATIONSHIP WITH THE TOTAL ACTUATION FORCE

As shown in Fig. S8(a), P(B) and G_{\min} exhibit a linear relationship, where G_{\min} decreases linearly as P(B) increases. Consequently, this linearity ensures that the sum of $P(B) + G_{\min}$ also follows a linear trend as we change P(B). Notably, the value of $P(B) + G_{\min}$ will not exceed 0.38, because the symmetric bistable system under quasistatic loading, a dimensionless force of 0.38 is required for switching between stable states. Although the "total actuation force" increases with P(B), and Fig. S8(b) may give the impression that the static bias P(B) does not offer any advantage, it is important to note, as shown in Fig. 4(b) of the main text, that a larger static P(B) results in a larger non-fractal region for switching behavior. This reduction in uncertainty could be significant for experimental studies and the design of future magnetically responsive morphing metamaterials.



FIG. S8. (a) Linear relationship between the minimum dynamic forcing amplitude (G_{\min}) and the applied static bias force $(P(B), G_{\min})$ represents the minimum forcing amplitude required to switch between stable states in the parameter space shown in Figs. S2 to S4. (b) Total actuation force $(P(B) + G_{\min})$ as a function of P(B), illustrating a linear trend. Notably, the value of $P(B)+G_{\min}$ does not exceed 0.38, because the symmetric bistable system under quasistatic loading, a dimensionless force of 0.38 is required for switching between stable states.

VI. BIDIRECTIONAL SWITCHING BEHAVIOR BY REVERSING THE STATIC BIAS FORCE

We also conduct time-domain simulations to investigate bidirectional switching by reversing the direction of the static bias force. For P(B) = 0.100, the system switches from the stable state $u_{-1} = -1$ to the stable state $u_{+1} = +1$ under the parameters $(G, \Omega) = (0.125, 1.209412)$ and a damping ratio of $\gamma = 0.07$, as shown in Fig. 3(a). By reversing the static bias force to P(B) = -0.100 while keeping the same parameters of the forcing amplitude, frequency, and damping ratio, the system achieves the reverse transition from the stable state $u_{+1} = +1$ to the stable state $u_{-1} = -1$, as shown in Fig. S9. This confirms that the direction of the static bias force enables bidirectional switching between the two stable states of a bistable buckled beam.



FIG. S9. Bidirectional switching of the bistable buckled beam by reversing the static bias force direction. For P(B) = 0.100, the system transitions from the $u_{-1} = -1$ stable state to the $u_{+1} = +1$ stable state under the parameters $(G, \Omega) = (0.125, 1.209412)$ and $\gamma = 0.07$, as shown in Fig. 3(a) of the main manuscript. Reversing the static bias force to P(B) = -0.100 enables the reverse transition from $u_{+1} = +1$ to $u_{-1} = -1$, demonstrating the bidirectional switching behavior of the system.